

**A natural family of immersed Lagrangian deformations of
a branched covering of a special Lagrangian 3-sphere in a Calabi-Yau 3-fold
and its deviation from Joyce's criteria:
Potential image-support rigidity of A-branes that wrap around a sL S^3**

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Abstract

Using a hyperKähler rotation on complex structures of a Calabi-Yau 2-fold and rolling of an isotropic 2-submanifold in a symplectic 6-manifold, we construct, by gluing, a natural family of immersed Lagrangian deformations of a branched covering of a special Lagrangian 3-sphere in a Calabi-Yau 3-fold and study how they deviate from being deformable to a family of special Lagrangian deformations by examining in detail Joyce's criteria on this family. The result suggests a potential image-support rigidity of A-branes that wrap around a special Lagrangian 3-sphere in a Calabi-Yau 3-fold, which resembles a similar phenomenon for holomorphic curves that wrap around a rigid smooth rational curve in a Calabi-Yau 3-fold in Gromov-Witten theory.

Key words: A-brane, wrapping; special Lagrangian 3-sphere, branched covering; Joyce's criteria, immersed Lagrangian deformation.

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Chien-Hao Liu dedicates this note to professors (time-ordered)
Hai-Chau Chang^{}, Su-Win Yang, Ai-Nung Wang,*
who introduced him the beauty of modern geometry and topology
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^{*}(From C.-H.L.) Special tribute to *Prof. Chang*, a mentor and a friend throughout my college years who has changed the course of my life.

0. Introduction and outline.

In the geometric phase of the Wilson's theory-space for a boundary conformal field theory with weak D-brane tension, D-branes are realized in part as morphisms from Azumaya noncommutative spaces with a fundamental module (with a connection) to string-theory target-space(-time). For physical A-branes in the supersymmetric case, the connection is required to be flat, possibly with singularities, and the associated maps from the surrogates are required to be special Lagrangian morphisms. See [L-Y2: Sec. 2.1] for more explanations and references.

In the special case when D-branes wraps a special Lagrangian 3-sphere in a Calabi-Yau 3-fold, it follows from Robert McLean [McL] that the latter is rigid for a topological reason and it is natural to ask whether through such wrapping one can deform the image D-brane away from the given special Lagrangian 3-sphere. The answer would influence, for example, the details of the quiver gauge field theory associated to a collection of special Lagrangian 3-spheres that intersect transversely in a Calabi-Yau 3-fold and the multiple cover formula for D3-branes (or Euclidean D2-branes) that wrap a special Lagrangian 3-sphere.

Before we can address how to deal with this problem, we devote first this note to analyzing in detail how some existing constructions and techniques could fail in the current situation.

This is a sequel to [L-Y1] (D(6)) and [L-Y2] (D(7)). In the first part of this note (Sec. 1 – Sec. 3), we take a fundamental existence theorem of Dominic Joyce ([Jo3: III. Theorem 5.3]) as the starting point (cf. Sec. 1), construct a natural family $f^t : N^t \rightarrow Y$ of smooth immersed Lagrangian submanifolds – in a similar spirit as is done in [Sa1] of Sema Salur for resolving a codimension-2 singularity of a singular special Lagrangian submanifold in a Calabi-Yau 3-fold – that has the special Lagrangian branched covering $f : X \rightarrow Y$ to begin with as its limit in C^∞ -topology as well as in the sense of current when $t \rightarrow 0$ (cf. Sec. 3.1), and check how Joyce's criteria of deformability to special Lagrangian submanifolds and standard techniques to justify them behave on the family $\{f^t\}_t$ (cf. Sec. 3.2 – Sec. 3.4). Some necessary background and ingredients for the study are given in Sec. 2 and the beginning of Sec. 3.2; and a summary on the deviation of $\{f_t\}_t$ from Joyce's criteria is given in Sec. 4. The investigation reveals a potential image-support rigidity of A-branes that wrap around a special Lagrangian 3-sphere in a Calabi-Yau 3-fold. This resembles a similar phenomenon for holomorphic curves that wrap around a rigid smooth rational curve in a Calabi-Yau 3-fold in Gromov-Witten theory.

Finally, we should remark that, instead of taking Joyce's Existence Theorem as the starting point, one can also proceed to understand the problem from a direct approach through a route following Adrian Butscher ([Bu1], [Bu2]), Yng-Ing Lee [Lee], and Semar Salur [Sa1].

Convention. Standard notations, terminology, operations, facts¹ in (1) Riemannian, spectral/hyperKähler geometry; (2) analysis on Riemannian manifolds; (3) symplectic/calibrated geometry; (4) branched coverings can be found respectively in (1) [G-H-L], [S-Y]/[Jo2]; (2) [Au]; (3) [McD-S], [G-S]/[H-L], [Ha], [McL]; (4) [Ro].

- ‘*n*-(sub)manifold’ for *real* (sub)manifold of (real) dimension n vs. ‘*n*-fold’ for *complex* manifold of (complex) dimension n .
- ‘Branch locus’ Γ of a map vs. ‘graph’ $\Gamma(\alpha)$ of a 1-form α vs. the space $\Gamma(\cdot)$ of *sections* of a bundle vs. the ‘Christoffel symbols’ Γ_{jk}^i .
- ‘Connection’ ω and ‘curvature’ Ω vs. ‘Kähler/symplectic structure’ ω and ‘holomorphic n -form’ Ω . The latter vs. the space $\Omega^k(N)$ of ‘ k -forms’ on a manifold N .

¹Cf. [L-Y2: footnote 2] (D(7)): *Apology*.

- ‘*Distribution*’ in the sense of generalized functions vs. ‘*distribution*’ in the sense of a sub-bundle of a tangent bundle or its restriction to a submanifold.
- The various constants C, D, \dots that appear in an estimate are unspecified constants that, in general, may of different values at difference places.
- The various built-in *projection maps* of bundles to their base are denoted by π .
- The Laplacian Δ and its eigenvalues λ_k for a Riemannian manifold (M, g) . When $\partial M \neq \emptyset$, this is referred to the related Dirichlet problem with vanishing boundary value.
- Notations related to the various constructions follow their counterpart in [Jo3: III and IV] as much as we can. Proofs that follow essentially the same argument as in their counterpart in ibidem will be either omitted or given only a sketch for spelling out the modified part.
- A partial review of D-branes and Azumaya noncommutative geometry is given in [L-Y1] (D(6)). The current work addresses [L-Y2: Sec. 2.3, Question 2.3.9] (D(7)).

Outline.

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1 Joyce's Existence Theorem on immersed special Lagrangian submanifolds.

For the basic definitions, notations, and terminology to be used in this work, we recall Joyce's Existence Theorem on deforming an immersed almost special Lagrangian submanifold to an immersed special Lagrangian submanifold, ([Jo3: III. Sec. 5.1 and Sec. 5.2])² with only mild adaptation from the general almost Calabi-Yau case to the Calabi-Yau case and mild change of notations for consistency with later part of the work.

Let (N, g) be a Riemannian manifold, with injective radius $\delta(g)$ and volume form dV_g . Denote the various Banach spaces of functions on N as follows:

- $C^k(N)$, $k \geq 0$: the *Banach space* of continuous bounded functions on N that have k continuous bounded derivatives; $\|f\|_{C^k} := \sum_{j=0}^k \sup_N |\nabla^j f|$. $C^\infty(N) := \bigcap_{k \geq 0} C^k(N)$.
- $C^{k,\alpha}$, $k \geq 0$, $\alpha \in (0, 1)$: the *Hölder space* of elements $f \in C^k(N)$ for which $|\nabla^k f|_\alpha := \sup_{x \neq y, \text{dist}(x,y) < \delta(g)} \frac{|\nabla^k f(x) - \nabla^k f(y)|}{d(x,y)^\alpha}$ is finite; $\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + |\nabla^k f|_\alpha$.
- $L^q(N)$, $q \geq 1$: the *Lebesgue space* of locally integrable functions f on N for which the norm $\|f\|_{L^q} := \left(\int_N |f|^q dV_g \right)^{1/q}$ is finite.
- $L_k^q(N)$, $q \geq 1$, $k \geq 0$: the *Sobolev space* of elements $f \in L^q(N)$ such that f is k times weakly differentiable and $|\nabla^j f| \in L^q(N)$ for $j \leq k$; $\|f\|_{L_k^q} := \left(\sum_{j=0}^k \int_N |\nabla^j f|^q \right)^{1/q}$.

Theorem 1.1. [Sobolev embedding]. ([Au: Theorem 2.30], [Jo3: III. Theorem 5.1].) *Suppose (N, g) is a compact Riemannian n -manifold, $k \geq l \geq 0$ are integers, $\alpha \in (0, 1)$, and $q, r \geq 1$. If $\frac{1}{q} \leq \frac{1}{r} + \frac{k-l}{n}$, then $L_k^q(N)$ is continuously embedded in $L_l^r(N)$ by inclusion. If $\frac{1}{q} \leq \frac{k-l-\alpha}{n}$, then $L_k^q(N)$ is continuously embedded in $C^{l,\alpha}(N)$ by inclusion.*

Definition 1.2. [basic setup]. ([Jo3: III. Definition 5.2].) Let (M, J, ω, Ω) be a Calabi-Yau m -fold with metric g_M . Let N be a compact, oriented, immersed, Lagrangian m -submanifold in M , with immersion $\iota : N \rightarrow M$, so that $\iota^* \omega \equiv 0$. Define $g := \iota^* g_M$, so that (N, g) is a Riemannian manifold.

- (1) *Phase function $e^{i\theta}$.* Let $dV := dV_g$ be the volume form on N induced by the metric g and orientation. Then $|\iota^* \Omega| \equiv 1$, calculating $|\cdot|$ using g on N . Therefore, we may write

$$\iota^* \Omega = e^{i\theta} dV \quad \text{on } N,$$

for some phase function $e^{i\theta}$ on N . Suppose that $\cos \theta \geq \frac{1}{2}$ on N . Then we can choose θ to be a smooth function $\theta : N \rightarrow (-\frac{\pi}{3}, \frac{\pi}{3})$. Suppose that $[\iota^*(Im \Omega)] = 0$ in $H^m(N; \mathbb{R})$. Then $\int_N \sin \theta dV = 0$.

- (2) *Vector space W .* Let $W \subset C^\infty(N)$ be a given a finite-dimensional vector space with $1 \in W$. Define $\pi_W : L^2(N) \rightarrow W$ be the projection onto W using the L^2 -inner product.
- (3) *Neighborhood \mathcal{B}_r of the zero-section in T^*N .* For $r > 0$, define $\mathcal{B}_r \subset T^*N$ to be the bundle of 1-forms α on N with $|\alpha| < r$. \mathcal{B}_r is a noncompact $2m$ -manifold with natural projection $\pi : \mathcal{B}_r \rightarrow N$, whose fiber at $x \in N$ is the ball of radius r about 0 in T_x^*N . We identify N also with the zero-section of \mathcal{B}_r and write $N \subset \mathcal{B}_r$.

²While it is theorems in the embedded case that are stated in Joyce's work, they generalize immediately to the immersed case.

- (4) *Geometry on \mathcal{B}_r .* At each $y \in \mathcal{B}_r$ with $\pi(y) = x \in N$, the Levi-Civita connection ∇ of g on T^*N defines a splitting $T_y\mathcal{B}_r = H_y \oplus V_y$ into horizontal and vertical subspaces H_y, V_y , with $H_y \simeq T_xN$ and $V_y \simeq T_x^*N$. Let $\hat{\omega} = \omega_{can}$ for the canonical symplectic structure on $\mathcal{B}_r \subset T^*N$, defined using $T\mathcal{B}_r = H \oplus V$ and $H \simeq V^*$. Define a natural Riemannian metric \hat{g} on \mathcal{B}_r such that the subbundles H, V are orthogonal, and $\hat{g}|_H = \pi^*(g)$, $\hat{g}|_V = \pi^*(g^{-1})$.

Let $\hat{\nabla}$ be the connection on $T_*\mathcal{B}_r \simeq H \oplus V$ given by the lift of the Levi-Civita connection ∇ of g on N in the horizontal directions H , and by partial differentiation in the vertical directions V , which is well-defined as $T_*\mathcal{B}_r$ is naturally trivial along each fiber.³ Then $\hat{\nabla}$ preserves \hat{g} , $\hat{\omega}$, and the splitting $T_*\mathcal{B}_r = H \oplus V$. It is not torsion-free in general, but has torsion $T(\hat{\nabla})$ depending linearly on the Riemann curvature $R(g)$.

For convenience and with a slight abuse of terminology, we will call $\hat{\nabla}$ the *pull-back connection* in the bundle $T_*\mathcal{B}_r$ (or $T_*(T^*N)$) of the Levi-Civita connection ∇ in T^*N via $\pi : \mathcal{B}_r \rightarrow N$. (Cf. Sec. 2.4.)

- (5) *\mathcal{B}_r as immersed Lagrangian neighborhood and m -form β on \mathcal{B}_r* Since ι is an immersed Lagrangian submanifold, it follows from the Immersed Lagrangian Neighborhood Theorem that for some small $r > 0$, there exists an immersion $\Phi : \mathcal{B}_r \rightarrow M$ such that $\Phi^*\omega = \hat{\omega}$ and $\Phi|_N = \iota$. Define an m -form β on \mathcal{B}_r by $\beta = \Phi^*(Im\Omega)$.
- (6) *Graph of 1-forms.* If $\alpha \in C^\infty(T^*N)$ with $|\alpha| < r$, write $\Gamma(\alpha)$ for the *graph* of α in \mathcal{B}_r . Then $\Phi_* : \Gamma(\alpha) \rightarrow M$ is a compact immersed submanifold in M homotopic to $\iota \circ \pi|_{\Gamma(\alpha)}$.

Theorem 1.3. [Joyce: from almost sL to sL]. ([Jo3: III. Theorem 5.3].) *With the above notations with $m \geq 3$, let $\kappa > 1$ and $A_1, A_2, A_4, A_5, A_6, A_7, A_8$ be real.⁴ Then there exist $\epsilon, K > 0$ depending only on κ, A_1, \dots, A_8 and m such that the following holds:*

Suppose $0 < t \leq \epsilon$ and Definition 0.2 holds with $r = A_1 t$, and

- (i) $\|\sin\theta\|_{L^{2m/(m+2)}} \leq A_2 t^{\kappa+m/2}$, $\|\sin\theta\|_{C^0} \leq A_2 t^{\kappa-1}$, $\|d(\sin\theta)\|_{L^{2m}} \leq A_2 t^{\kappa-3/2}$, and $\|\pi_W(\sin\theta)\|_{L^1} \leq A_2 t^{\kappa+m-1}$.
- (iii) $\|\hat{\nabla}^k \beta\|_{C^0} \leq A_4 t^{-k}$ for $k = 0, 1, 2$, and 3.
- (iv) *The injective radius $\delta(g)$ satisfies $\|\delta(g)\| \geq A_5 t$.*
- (v) *The Riemann curvature $R(g)$ satisfies $\|R(g)\|_{C^0} \leq A_6 t^{-2}$.*
- (vi) *If $v \in L_1^2(N)$ with $\pi_W(v) = 0$, then $v \in L^{2m/(m-2)}(N)$ by Theorem 0.1, and $\|v\|_{L^{2m/(m-2)}} \leq A_7 \|dv\|_{L^2}$.*
- (vii) *For all $w \in W$, we have $\|d^*dw\|_{L^{2m/(m+2)}} \leq \frac{1}{2} A_7^{-1} \|dw\|_{L^2}$. For all $w \in W$ with $\int_N w dV = 0$, we have $\|w\|_{C^0} \leq A_8 t^{1-m/2} \|dw\|_{L^2}$.*

Here norms are computed using the metric g on N in (i), (v), (vi) and (vii), and the metric \hat{g} on $\mathcal{B}_{A_1 t}$ in (iii). Then there exists $f \in C^\infty(N)$ with $\int_N f dV = 0$ such that $\|df\|_{C^0} \leq K t^\kappa < A_1 t$ and $\Phi_ : \tilde{N} := \Gamma(df) \rightarrow M$ is an immersed special Lagrangian m -manifold in (M, J, ω, Ω) .*

See [Jo3: IV, last paragraph of Sec. 5.1] for the reasons behind the design of the theorem.

³Recall the projection map $\pi : \mathcal{B}_r \rightarrow N$. The Levi-Civita ∇ on T^*N induces a bundle inclusion $\pi^*(T^*N) \hookrightarrow T_*\mathcal{B}_r$ with image H . The pull-back partial connection on $\pi^*(T^*N)$ gives then a partial connection $\hat{\nabla}_H$ on H along the horizontal distribution on \mathcal{B}_r associated to ∇ . $V = \text{Ker}(\pi_* : T_*\mathcal{B}_r \rightarrow T_*N)$ with a built-in flat partial connection $\hat{\nabla}_V$ along fibers of π . As smooth vector bundles, $T_*\mathcal{B}_r = H \oplus V$ and the direct sum $\hat{\nabla}_H \oplus \hat{\nabla}_V$ of partial connections on the direct summands gives the connection $\hat{\nabla}$ on $T_*\mathcal{B}_r$.

⁴For an almost Calabi-Yau m -fold (M, J, ω, Ω) , $\psi^{2m} \omega^m / m! = (-1)^{m(m-1)/2} (i/2)^m \Omega \wedge \bar{\Omega}$ for a unique smooth function $\psi : M \rightarrow (0, \infty)$. In our case, $\psi \equiv 1$, $A_3 = 1$, and Condition (ii) of [Jo3: III Theorem 5.3], which states that $\psi \geq A_3$ on N , holds automatically. As this work is based upon Joyce's work, we maintain the original notation and labelling of conditions in Joyce's Theorem.

2 Basic Riemannian, complex, and symplectic ingredients.

Preliminary ingredients that are needed to apply Joyce's Theorem in our situation are collected in this section. They follow from standard techniques in Riemannian, complex, and symplectic geometry.

2.1 A lower bound of the first eigenvalue of the Laplacian on a Riemannian manifold with tame curvature singularity.

Definition 2.1.1. [Riemannian metric with tame curvature singularity]. Given a closed smooth manifold X , let g' be a Riemannian metric defined on a dense open submanifold X' of X with the property that

- (1) g' does not extend to a Riemannian metric on X ,
- (2) there exist a Riemannian metric g on X and a constant $c > 1$ such that

$$\frac{1}{c^2} g \leq g' \leq c^2 g \quad \text{on } X'.$$

Let $Z := X - X'$. We say that g' is a *Riemannian metric on X with tame curvature singularity* supported on Z . In other words, while g' doesn't extend to X , it admits a quasi-conformal deformation with uniformly bounded dilatation that extends to a Riemannian metric on X . We call (X, g') a closed *Riemannian manifold with tame curvature singularity*.

The following lemma is the counterpart of [Jo3: I. Theorem 2.17] in our situation.

Lemma 2.1.2. [lower bound for first eigenvalue λ_1 of Laplacian]. *Let (X, g') be a (connected) closed Riemannian m -manifold with tame curvature singularity and X' be a (connected) dense open submanifold of X on which g' is defined and that there exists an exhausting sequence $X'_1 \subset X'_2 \subset \dots \subset X'$ of (connected) embedded compact submanifolds-with-smooth-boundary of dimension m with $\bigcup_{i=1}^{\infty} X'_i = X'$. Then there exists a constant $C > 0$ such that whenever $u \in C_{cs}^2(X')$ with $\int_{X'} u dV_{g'} = 0$, one has*

$$\|u\|_{L^2} \leq C \|du\|_{L^2} \leq C^2 \|\Delta u\|_{L^2}.$$

Proof. Let g be a Riemannian metric on X such that $\frac{1}{c^2} g \leq g' \leq c^2 g$ on X' for some $c > 1$. Then it follows from the Minimax Principle for the eigenvalues of Laplacians that

$$0 < \frac{1}{c^{2m+2}} \lambda_{k,(X,g)} \leq \frac{1}{c^{2m+2}} \lambda_{k,(X'_i,g)} \leq \lambda_{k,(X'_i,g')},$$

for all $k \geq 1$ and $i \geq 1$. This gives in particular a positive uniform lower bound $\lambda_{1,(X,g)}/c^{2m+2}$ for all the first eigenvalues $\lambda_{1,(X'_i,g')}$ of the Laplacians on (X'_i, g') . Since $C_{cs}^2(X') = \bigcup_{i=1}^{\infty} C_{cs}^2(X'_i)$, as done in [Jo3: I. Proof of Theorem 2.17, last two paragraphs], taking the set of normalized eigenfunctions of the Laplacian $\Delta_{(X'_i,g')}$ as an orthonormal basis in the Hilbert space V_i from completing $\left\{ u \in C_{cs}^2(X'_i) : \int_{X'_i} u = 0 \right\}$ in the L^2 -norm gives then the inequalities in Lemma with $C = c^{m+1} / \lambda_{1,(X,g)}^{1/2}$. This completes the proof. \square

Example 2.1.3. [branched covering of S^3]. Let (S^3, g_0) be a Riemannian 3-sphere and $f : X \rightarrow S^3$ be a (connected smooth) branched covering of S^3 of finite degree. Assume that the branch locus $\underline{\Gamma} \subset S^3$ is a smooth link in S^3 and so is the branch locus $\Gamma := f^{-1}(\underline{\Gamma})$ in X , and that $\Gamma \simeq \underline{\Gamma}$ under f .

(a) *Claim.* (X, f^*g_0) is a Riemannian manifold with tame curvature singularity supported on Γ .

Proof. Since any two Riemannian metrics on a closed smooth manifold are quasi-conformal with uniformly bounded dilation, without loss of generality and through a partition of unity we may assume that the metric g_0 on S^3 has the additional property that there exists a tubular neighborhood $N_\epsilon(\underline{\Gamma})$ of $\underline{\Gamma} \subset S^3$ such that $g_0|_{N_\epsilon(\underline{\Gamma})}$ is isometric to a finite disjoint union $\coprod_{i \in H_0(\underline{\Gamma}; \mathbb{Z})} \underline{N}_i$ of the solid torus $\underline{N}_i \simeq D_\epsilon^2 \times S^1$ with the flat product metric. Here, D_ϵ^2 is a closed disk at the origin of radius ϵ in the standard flat \mathbb{R}^2 . Each connected component N_i , $i \in H_0(\Gamma; \mathbb{Z})$, of $f^{-1}(N_\epsilon(\underline{\Gamma}))$ is diffeomorphic to a solid torus $D^2 \times S^1$ as well. With an appropriate choice of local coordinates on $N_i - \Gamma$, we may assume that

$$\begin{aligned} f : N_i - \Gamma &\longrightarrow \underline{N}_i \\ (r, \phi, \theta) &\longmapsto (\underline{r}, \underline{\phi}, \underline{\theta}) = (r, m_i \phi, \theta) \end{aligned} ,$$

where $m_i \geq 2$ is the branching index of f along $\Gamma \cap N_i$; (r, ϕ) , $(\underline{r}, \underline{\phi})$ are the polar coordinates of (the flat) D^2 , D_ϵ^2 respectively; and $\theta \in \mathbb{R}/2\pi \simeq S^1$. In terms of this, $g_0|_{\underline{N}_i}$ is given by $ds_0^2 = d\underline{r}^2 + \underline{r}^2 d\underline{\phi}^2 + \frac{l_i^2}{4\pi^2} d\underline{\theta}^2$, where l_i is the length of $\underline{\Gamma} \cap \underline{N}_i$ in (S^3, g_0) , and $(f^*g_0)|_{N_i - \Gamma}$ is given by

$$ds'^2 = dr^2 + m_i^2 r^2 d\phi^2 + \frac{l_i^2}{4\pi^2} d\theta^2 = (dr)^2 + (m_i r d\phi)^2 + \left(\frac{l_i}{2\pi} d\theta \right)^2 =: \omega_r^2 + \omega_\phi^2 + \omega_\theta^2.$$

Let $h_i : (0, \epsilon) \rightarrow \mathbb{R}^+$ be an increasing smooth function with $h|_{(0, \epsilon/3)}$ the constant $1/m_i$ and $h|_{(2\epsilon/3, \epsilon)}$ the constant 1. This defines a smooth function, still denoted by h_i , on $N_i - \Gamma$ by identifying $(0, \epsilon)$ as the r -coordinate. Let $g' := f^*g_0$ on $X' := X - \Gamma$ and g be the metric on X' defined by

$$ds^2 = \begin{cases} g' & \text{on } X' - N_\epsilon(\Gamma), \\ \omega_r^2 + (h_i \cdot \omega_\phi)^2 + \omega_\theta^2 & \text{on } N_i - \Gamma, \ i \in H_0(\Gamma; \mathbb{Z}). \end{cases}$$

Let $c := \max_i \{m_i\}$. Then, by construction, $\frac{1}{c^2}g \leq g \leq g' \leq c^2g$ on X' . Furthermore, g on X' extends to a Riemannian metric on X without singularity. This proves the claim. \square

(b) Define $X'_j := \{p \in X' : \text{distance}_{g'}(p, \Gamma) \geq 1/(j + j_0)\}$, $j \in \mathbb{N}$. For $j_0 \in \mathbb{N}$ large enough, $X'_1 \subset X'_2 \subset \dots$ are connected compact 3-manifolds-with-smooth-boundary that exhaust X' : $\bigcup_j X'_j = X'$. It follows thus from Claim in Part (a) and Lemma 2.1.2 that:

- There exists a constant $C > 0$ such that whenever $u \in C_{cs}^2(X')$ with $\int_{X'} u dV_{g'} = 0$, one has $\|u\|_{L^2} \leq C \|du\|_{L^2} \leq C^2 \|\Delta u\|_{L^2}$.

2.2 A class of embedded special Lagrangian submanifolds in the flat Calabi-Yau 3-fold $\mathbb{R}^2 \times S^1 \times \mathbb{R}^3$.

Let $Y' := \mathbb{R}^2 \times S^1 \times \mathbb{R}^3$ be the flat Calabi-Yau 3-fold with

- the real coordinates $(u_1, u_2, u_3, v_1, v_2, v_3)$, where $u_1, u_2, v_1, v_2, v_3 \in \mathbb{R}$ and $u_3 \in \mathbb{R}/l$,

- the complex structure J' specified by the complex coordinates (z_1, z_2, z_3) with $z_1 = u_1 + \sqrt{-1}v_1$, $z_2 = u_2 + \sqrt{-1}v_2$, and $z_3 = u_3 + \sqrt{-1}v_3$,
- the Kähler form $\omega' = \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3)$, which specifies the Kähler metric $g' = |dz_1|^2 + |dz_2|^2 + |dz_3|^2$, and
- the holomorphic 3-form $\Omega' = dz_1 \wedge dz_2 \wedge dz_3$, which gives the calibration $Re \Omega'$.

We'll denote $(Y', J', \omega', \Omega')$ also collectively by Y' .

A class of embedded special Lagrangian submanifolds of Y' via rolling isotropic submanifolds.

As Calabi-Yau manifolds, Y' is isomorphic to the product $Y'' \times Y''' := \mathbb{R}^4 \times (S^1 \times \mathbb{R})$ of a Calabi-Yau 2-fold and a Calabi-Yau 1-fold, where \mathbb{R}^4 is the (u_1, u_2, v_1, v_2) -coordinate subspace and $S^1 \times \mathbb{R}^1$ is the (u_3, v_3) -coordinate subspace, with the induced Calabi-Yau manifold structures $(J'', \omega'', \Omega'')$ and $(J''', \omega''', \Omega''')$ respectively. Note that Y'' is hyperKähler. Thus, special Lagrangian submanifolds in Y'' can be obtained by introducing a new complex structure \hat{J}'' on Y'' – with the associated complex coordinates given by $(\hat{z}_1, \hat{z}_2) = (u_1 + \sqrt{-1}u_2, v_1 - \sqrt{-1}v_2)$ – via a hyperKähler rotation and taking smooth holomorphic curves C in $\hat{Y}'' := (Y'', \hat{J}'')$. Such C 's are isotropic submanifolds in $Y'' \times Y'''$ (with the original complex structure (J'', J''')). $C \times S^1 \times \{a\}$, for $a \in \mathbb{R}$, give then a class of embedded special Lagrangian submanifolds in $Y'' \times Y'''$ and, hence, in Y' .

Remark 2.2.1. [basic expression under hyperKähler rotation]. For later use, note that in terms of the hyperKähler rotated complex coordinates (\hat{z}_1, \hat{z}_2) on \mathbb{R}^4 ,

$$\begin{aligned} \frac{\sqrt{-1}}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2) &= \frac{1}{2}(d\hat{z}_1 \wedge d\hat{z}_2 + d\bar{\hat{z}}_1 \wedge d\bar{\hat{z}}_2) = Re(d\hat{z}_1 \wedge d\hat{z}_2), \\ dz_1 \wedge dz_2 &= \frac{\sqrt{-1}}{2}(d\hat{z}_1 \wedge d\bar{\hat{z}}_1 + d\hat{z}_2 \wedge d\bar{\hat{z}}_2) - \frac{1}{2}(d\hat{z}_1 \wedge d\hat{z}_2 - d\bar{\hat{z}}_1 \wedge d\bar{\hat{z}}_2) \\ &= -\frac{1}{2}Im(d\hat{z}_1 \wedge d\bar{\hat{z}}_1 + d\hat{z}_2 \wedge d\bar{\hat{z}}_2) - \sqrt{-1}Im(d\hat{z}_1 \wedge d\hat{z}_2). \end{aligned}$$

Deformations of a branched covering of the special Lagrangian $\mathbb{R}^2 \times S^1 \times \{0\}$ to embedded special Lagrangian submanifolds of Y' .

Consider the embedded special Lagrangian submanifold $L' := \mathbb{R}^2 \times S^1 \times \{0\}$, where $0 \in \mathbb{R}^3$ is the origin, of Y' . In terms of the \mathbb{C} - \mathbb{R} -valued coordinates $(\hat{z}_1, u_3, \hat{z}_2, v_3)$ on Y' (and hence on L' as well), let

$$\begin{aligned} \psi : L \simeq \mathbb{R}^2 \times S^1 &\longrightarrow Y' \\ (x_1, x_2, x_3) &\longmapsto ((x_1 + \sqrt{-1}x_2)^m, x_3, 0, 0) \end{aligned}$$

be a branched covering of L' of degree $m \geq 2$ with branch locus $\Gamma = \{(0, 0)\} \times S^1 \subset L$ and $\Gamma' = \{0\} \times S^1 \times \{(0, 0)\} \subset L'$ respectively. This is an immersion in codimension 1 (cf. [L-Y6: Definition 2.3.8] (D(7))). It can be deformed to smooth special Lagrangian embeddings by⁵

$$\begin{aligned} \psi^a : L \simeq \mathbb{R}^2 \times S^1 &\longrightarrow Y' \\ (x_1, x_2, x_3) &\longmapsto ((x_1 + \sqrt{-1}x_2)^m, x_3, a^{1/m}(x_1 + \sqrt{-1}x_2), 0), \end{aligned}$$

⁵ ψ^a can be defined alternatively by $(x_1, x_2, x_3) \longrightarrow (a^{-1/m}(x_1 + \sqrt{-1}x_2)^m, x_3, (x_1 + \sqrt{-1}x_2), 0)$. This has the same image special Lagrangian submanifold L'^a but now with $D\psi^a(0, 0, x_3)$ independent of a . While even the C^0 -convergence of the alternative ψ^a to ψ fails for ψ^a thus alternatively defined, $\psi^a \rightarrow \psi$, as $a \rightarrow 0$, remains to hold in the sense of currents.

for $a > 0$. ψ^a embeds L into Y' as the smooth embedded special Lagrangian submanifold

$$L'^a := \{a\hat{z}_1 - \hat{z}_2^m = 0, v_3 = 0\} \subset Y'.$$

Then as $a \rightarrow 0$, ψ^a is C^∞ -convergent to $\psi =: \psi^0$ on any compact subset L . One can also interpret this convergence in the sense of currents on Y' . (In notation, $L'^a \rightarrow mL'$ as $a \rightarrow 0$.)

Let $L'^\diamond := L' \cap \{\hat{z}_1 \neq 0\}$, $L'^{a,\diamond} := L'^a \cap \{\hat{z}_1 \neq 0\}$ for $a > 0$, $L^\diamond := \{(x_1, x_2) \neq (0, 0)\} (\simeq (\mathbb{R}^2 - \{(0, 0)\}) \times S^1) \subset L$, and $Y'^\diamond := \{\hat{z}_1 \neq 0\} (\simeq (\mathbb{R}^2 - \{(0, 0)\}) \times S^1 \times \mathbb{R}^3) \subset Y'$ with the induced Calabi-Yau structure still denoted by (J', ω', Ω') . Then $L'^{a,\diamond}$ is an open dense subset of L'^a that can be expressed via the graph of an exact 1-form on L^\diamond as follows. First, observe that since $\omega' = du_1 \wedge dv_1 + du_2 \wedge dv_2 + du_3 \wedge dv_3$, the map

$$\begin{aligned} \Phi' : \quad T^*L'^\diamond &\longrightarrow Y'^\diamond \\ (u_1, u_2, u_3, p_{u_1}, p_{u_2}, p_{u_3}) &\longmapsto (u_1 + \sqrt{-1}u_2, u_3, p_{u_1} - \sqrt{-1}p_{u_2}, p_{u_3}) \end{aligned}$$

where $(u_1, u_2, u_3, p_{u_1}, p_{u_2}, p_{u_3})$ corresponds to $p_{u_1}du_1 + p_{u_2}du_2 + p_{u_3}du_3 \in T^*_{(u_1, u_2, u_3)}L'^\diamond$, is a symplectomorphism with $\Phi'^*\omega' = \omega'_{can}$. Precomposing Φ' with the symplectic covering map $f_1 : T^*L^\diamond \rightarrow T^*L'^\diamond$ that is canonically induced by the covering map $f : L^\diamond \rightarrow L'^\diamond$, one obtains a covering map

$$\begin{aligned} \Phi &:= \Phi' \circ f_1 : T^*L^\diamond \longrightarrow Y'^\diamond \\ (x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) &\longmapsto ((x_1 + \sqrt{-1}x_2)^m, x_3, \frac{1}{m}(p_{x_1} - \sqrt{-1}p_{x_2})(x_1 + \sqrt{-1}x_2)^{1-m}, p_{x_3}) \end{aligned}$$

of degree m such that $\Phi^*\omega' = \omega_{can}$. By construction, $L'^{a,\diamond}$ lifts under Φ to a smooth section $\tilde{L}'^{a,\diamond}$ of T^*L^\diamond over L^\diamond that is Lagrangian with respect to $\omega_{can} = dx_1 \wedge dp_{x_1} + dx_2 \wedge dp_{x_2} + dx_3 \wedge dp_{x_3}$. Explicitly,

$$\begin{aligned} \Phi^{-1}(L'^{a,\diamond}) &= \left\{ (p_{x_1} - \sqrt{-1}p_{x_2}) - m a^{1/m} (x_1 + \sqrt{-1}x_2)^m e^{2\pi\sqrt{-1}k/m} = 0, \right. \\ &\quad \left. p_{x_3} = 0 \quad : k = 0, \dots, m-1 \right\} \end{aligned}$$

In the following, we choose $\tilde{L}'^{a,\diamond}$ that corresponds to the component with $k = 0$.

Lemma 2.2.2. *[$L'^{a,\diamond}$ via exact 1-form on L]. $\tilde{L}'^{a,\diamond}$ extends to a smooth section of T^*L that corresponds to the exact 1-form dh^a with $h^a := \frac{m}{m+1} a^{1/m} \operatorname{Re}((x_1 + \sqrt{-1}x_2)^{m+1}) \in C^\infty(L)$.*

Proof. Since $\tilde{L}'^{a,\diamond}$ is a smooth section of T^*L^\diamond that is Lagrangian with respect to ω_{can} , $\tilde{L}'^{a,\diamond}$ is the graph of a closed 1-form α on L^\diamond . To see that α is exact, one only needs to show that $\int_\gamma \alpha = 0$ for any (smooth) closed loop γ in L^\diamond . Since $H_1(L^\diamond; \mathbb{Z}) = \mathbb{Z}[\gamma_{(x_1, x_2)}] \oplus \mathbb{Z}[\gamma_{x_3}]$, where $\gamma_{(x_1, x_2)}$ is the (oriented) loop $\{x_1^2 + x_2^2 = \epsilon^2, x_3 = 0\}$ for a small $\epsilon > 0$ and γ_{x_3} is the (oriented) loop $\{x_1 = \epsilon, x_2 = 0\}$ in L^\diamond , and $\int_{\gamma_{x_3}} \alpha = 0$ due to the fact that $\tilde{L}'^{a,\diamond}$ lies in $\{p_{x_3} = 0\}$, we only need to check that $\int_{\gamma_{(x_1, x_2)}} \alpha = 0$. The latter follows immediately from the fact that, from the explicit equations for $\tilde{L}'^{a,\diamond}$, $\tilde{L}'^{a,\diamond}$ extends in T^*L to an embedded smooth Lagrangian submanifold that is realizable as a smooth section of T^*L and that $[\gamma_{(x_1, x_2)}] = 0$ in $H_1(L; \mathbb{Z})$.

Explicitly, let

$$\lambda_{can} = p_{x_1}dx_1 + p_{x_2}dx_2 + p_{x_3}dx_3 = \operatorname{Re}((p_{x_1} - \sqrt{-1}p_{x_2})(dx_1 + \sqrt{-1}dx_2)) + p_{x_3}dx_3$$

be the canonical 1-form on T^*L and $\pi : T^*L \rightarrow L$ be the projection map. Then

$$\begin{aligned} \alpha &= \pi_*(\lambda_{can}|_{\tilde{L}'^{a,\diamond}}) = \operatorname{Re} \left(m a^{1/m} (x_1 + \sqrt{-1}x_2)^m (dx_1 + \sqrt{-1}dx_2) \right) \\ &= \operatorname{Re} \left(\frac{m}{m+1} a^{1/m} d((x_1 + \sqrt{-1}x_2)^{m+1}) \right). \end{aligned}$$

This proves the lemma. \square

Notation 2.2.3. [partial scaling on Y']. The partial scaling $t \cdot (\hat{z}_1, u_3, \hat{z}_2, v_3) := (t\hat{z}_1, u_3, t\hat{z}_2, v_3)$ on Y' for $t > 0$ sends L'^a to $t \cdot L'^a := L'^{at^{m-1}}$. For convenience, we denote $\psi^{at^{m-1}}$ also by $t \cdot \psi^a$. In particular, given $a > 0$, then $t \cdot \psi^a \rightarrow \psi$ in the C^∞ -topology when $t \rightarrow 0$.

2.3 Lagrangian Neighborhood Theorems.

Recall first the following two Lagrangian Neighborhood Theorems:

Theorem 2.3.1. [immersed Lagrangian submanifold]. *Let (M, ω) be a symplectic manifold and $f : N \rightarrow M$ be a compact immersed Lagrangian submanifold. Then there exist a neighborhood $U \subset T^*N$ of the zero-section and an immersion $\Phi : U_N \rightarrow M$ such that $\Phi|_N = f$ and $\Phi^*\omega = \omega_{can}$, where ω_{can} is the canonical symplectic form on T^*N .*

Theorem 2.3.2. [Lagrangian foliation]. ([Jo3: I, Theorem 4.2] and [We: Theorem 7.1].) *Let (M, ω) be $2m$ -dimensional symplectic manifold and $N \subset M$ an embedded m -dimensional submanifold. Suppose $\{L_x : x \in N\}$ is a smooth family of embedded, noncompact Lagrangian submanifolds in M parameterized by $x \in N$ such that for each $x \in N$ we have $x \in L_x$ and $T_x L_x \cap T_x N = \{0\}$. Then there exist an open neighborhood U of the zero-section N in T^*N such that the fibers of the natural projection $\pi : U \rightarrow N$ are connected, and a unique embedding $\Phi : U \rightarrow M$ with $\Phi(\pi^{-1}(x)) \subset L_x$ for each $x \in N$, $\Phi|_N = id_N : N \rightarrow N$ and $\Phi^*\omega = \omega_{can} + \pi^*(\omega|_N)$, where ω_{can} is the canonical symplectic structure on T^*N .*

The first theorem follows from essentially the same proof as that for the embedded Lagrangian case ([McD-S] and [We]) and the second is stated in [Jo3: I, Theorem 4.2] as a variation of [We: Theorem 7.1].

Lagrangian Neighborhood Theorem for an embedded Lagrangian submanifold with a transverse Lagrangian distribution.

The following variation of Lagrangian Neighborhood Theorems is also needed in this work:

Theorem 2.3.3. [Lagrangian submanifold with transverse Lagrangian distribution]. *Let (M, ω) be a $2m$ -dimensional symplectic manifold, $N \subset M$ be a compact embedded Lagrangian submanifold, and $\Gamma_N \subset (T_*M)|_N$ be a distribution of tangent m -planes in M along N such that $\Gamma_x := \Gamma_N|_x$ is a Lagrangian subspace in $(T_x M, \omega_x)$ and that $\Gamma_x \cap T_x D = \{0\}$ for all $x \in D$. Then there exist an open neighborhood $U \subset T^*N$ of the zero-section and an embedding $\Phi : U \rightarrow M$ such that $\Phi|_N = id_N : N \rightarrow N$, $\Phi^*\omega = \omega_{can}$, where ω_{can} is the canonical symplectic form on T^*N , and $\Phi_*(T_0(T^*N)) = \Gamma_x$ for all $x \in N$.*

Proof. (a) Reformulation of the problem. Since the problem concerns only a neighborhood of N in M , by Theorem 2.3.1 one may assume that $(M, \omega) = (T^*N, \omega_{can})$ with N the zero-section. The fact that Γ_N is transverse to N implies that Γ_N defines a bundle map $T^*N \rightarrow T_*N$ and, hence, a bilinear map $(\cdot, \cdot)_{\Gamma_N} : T^*N \oplus T^*N \rightarrow \mathbb{R}$. The Lagrangian property of Γ_N implies that $(\cdot, \cdot)_{\Gamma_N}$ is a symmetric bilinear functional on T^*N .

(b) Realization of $(\cdot, \cdot)_{\Gamma_N}$ as a restriction of Hessian. Recall that a smooth function $f : T^*N \rightarrow \mathbb{R}$ such that $(df)|_N \equiv 0$ defines a symmetric bilinear functional $Hess_f$ on $(T_*(T^*N))|_N$

by setting $Hess_f(v, w) = v(\tilde{w}f)$, where $v, w \in T_x(T^*N)$ for some $x \in N$ and \tilde{w} is an extension of w to a smooth vector field in a neighborhood of x in T^*N . $Hess_f$ is independent of the extension \tilde{w} of w and, hence, well-defined. It is called the *Hessian of f at the critical manifold N of f* . Since T^*N is canonically embedded in $(T_*(T^*N))|_N$ as vector bundles over N , $Hess_f$ defines further a symmetric bilinear functional $Hess_f^\perp$ on T^*N by its restriction to T^*N .

Claim. *There exists a smooth function $f : T^*N \rightarrow \mathbb{R}$ that is supported in a compact neighborhood U' of the zero-section such that both $f|_N$ and $(df)|_N$ vanish and $Hess_f^\perp = (\cdot, \cdot)_{\Gamma_N}$. U' can be chosen to be arbitrarily small.*

Proof of Claim. Let $\pi : T^*N \rightarrow N$ be the built-in map. Then a local chart V (with coordinates $\mathbf{q} := (q_1, \dots, q_m)$) on N induces a local chart $\pi^{-1}(V)$ on T^*N (with induced canonical coordinates $(\mathbf{q}, \mathbf{p}) := (q_1, \dots, q_m, p_1, \dots, p_m)$). Let $\Gamma_V := \Gamma_N|_V$. Then the symmetric bilinear functional $(\cdot, \cdot)_{\Gamma_V}$ on T^*V can be expressed as a \mathbf{q} -dependent quadratic form $g_V := \frac{1}{2} \sum_{1 \leq i, j \leq m} a_{ij}(\mathbf{q}) p_i p_j$, where $a_{ij}(\mathbf{q}) = a_{ji}(\mathbf{q})$ for all i, j , in \mathbf{p} . As a function on $\pi^{-1}(V)$, g_V satisfies the property that both $g_V|_V$ and $(dg_V)|_V$ vanish and that $Hess_{g_V}^\perp = (\cdot, \cdot)_{\Gamma_V}$. Observe now the following property, which can be checked straightforwardly:

- For convenience, call a function $g : \pi^{-1}(V) \rightarrow \mathbb{R}$ *admissible* if g satisfies the condition that both $g|_V$ and $(dg)|_V$ vanish and that $Hess_g^\perp = (\cdot, \cdot)_{\Gamma_V}$. Then, if g_1, g_2 are admissible functions on T^*V and h_1, h_2 are positive functions on V , then $h_1 g_1 + h_2 g_2$ is an admissible function on T^*V .

Let $\{\mu_i\}_i$ be a partition of unity on N subordinate to a locally finite covering $\{V_i\}_i$ of N and g_{V_i} be an admissible function on T^*V_i , whose existence is demonstrated by the explicit example. Then it follows from the observation above that $g := \sum_i \mu_i g_{V_i}$ is an admissible function on T^*N . Finally, introduce a cutoff function $\chi : T^*N \rightarrow [0, 1]$ that is supported on an arbitrarily small neighborhood of the zero-section. Then $f := \chi g$ satisfies all the required properties in the Claim. \square

(c) Φ from time-1 map of Hamiltonian flow. Let $f : T^*N \rightarrow \mathbb{R}$ be a smooth function on T^*N as constructed in Part (b), X_f be the Hamiltonian vector field on T^*N associated to f (i.e. $i(X_f)\omega_{can} = df$) and $\Phi := \Phi_1 : T^*N \rightarrow T^*N$ be the time-1 map of the Hamiltonian flow Φ_t , $t \in \mathbb{R}$, on T^*N associated to f (i.e. $\frac{d}{dt}\Phi_t = X_f \circ \Phi_t$). By construction, Φ is a symplectomorphism on T^*N that leaves N fixed. Recall the proof of Claim in Part (b) and the notations and terminology therein. Then, since f is admissible,

$$f|_{\pi^{-1}(V)} = \frac{1}{2} \sum_{1 \leq i, j \leq m} a_{ij}(\mathbf{q}) p_i p_j + o(|\mathbf{p}|^2)$$

and

$$X_f|_{\pi^{-1}(V)} = \sum_{i=1}^m \left(\sum_{j=1}^m a_{ij}(\mathbf{q}) p_j \right) \frac{\partial}{\partial q_i} + o(|\mathbf{p}|).$$

Let $x \in V$, V_x be a neighborhood of x in $\pi^{-1}(V)$. Assume that V_x is small enough and let $\Phi'_{1, V_x} : V_x \rightarrow \pi^{-1}(V)$ be the time-1 map of the flow generated by $\sum_{i=1}^m (\sum_{j=1}^m a_{ij}(\mathbf{q}) p_j) \frac{\partial}{\partial q_i}$. Then, from the above approximation of $X_f|_{\pi^{-1}(V)}$,

$$\Phi_*(T_0(T^*N)) = \Phi'_{1, V_x, *}(T_0(T^*\pi^{-1}(V))) = \Gamma_x.$$

Take now $U = \Phi^{-1}(U')$. Then $\Phi : U \rightarrow M$ gives a neighborhood of N in M as required. This completes the proof of the theorem. \square

2.4 Admissible Lagrangian neighborhoods for L'^a in Y' under ψ^a and their geometry under partial scaling.

Recall the flat Calabi-Yau 3-fold $Y' = (\mathbb{R}^2 \times S^1 \times \mathbb{R}^3, J', \omega', \Omega')$ with the real (resp. complex, hyperKähler-rotated) coordinates $(u_1, u_2, u_3, v_1, v_2, v_3)$ (resp. $(z_1, z_2, z_3) = (u_1 + \sqrt{-1}v_1, u_2 + \sqrt{-1}v_2, u_3 + \sqrt{-1}v_3)$), $(\hat{z}_1, u_3, \hat{z}_2, v_3) = (u_1 + \sqrt{-1}u_2, u_3, v_1 - \sqrt{-1}v_2, v_3)$ and the embedded smooth special Lagrangian submanifolds $L' = \mathbb{R}^2 \times S^1 \times \{\mathbf{0}\}$ and $L'^a = \{a\hat{z}_1 - \hat{z}_2^m = 0, v_3 = 0\}$, for $a > 0$, in Y' in Sec. 2.2. Under the real coordinates, (Y', ω') is identified with (T^*L', ω_{can}) under the map $(u_1, u_2, u_3, v_1, v_2, v_3) = (u_1, u_2, u_3, p_{u_1}, p_{u_2}, p_{u_3})$. Recall also the embedded special Lagrangian map $\psi^a : L \rightarrow Y'$, $(x_1, x_2, x_3) \mapsto ((x_1 + \sqrt{-1}x_2)^m, x_3, a^{1/m}(x_1 + \sqrt{-1}x_2), 0)$, with image L'^a , $h^a \in C^\infty(L)$ in Lemma 2.2.2, and the partial scaling on Y' , $(\hat{z}_1, u_3, \hat{z}_2, v_3) \mapsto (t\hat{z}_1, u_3, t\hat{z}_2, v_3)$, for $t > 0$ in Notation 2.2.3.

Proposition 2.4.1. [admissible Lagrangian neighborhood: existence]. *Given $R'_0 > 0$, there exist a neighborhood U_L^a of the zero-section of T^*L and a symplectic embedding $\Phi_L^a : U_L^a \rightarrow Y'$ whose restriction to the zero-section is ψ^a such that*

$$\Phi_L^a|_{\pi^{-1}(\{|(x_1 + \sqrt{-1}x_2)^m| \geq R'_0\})}(\cdot) = \psi^a_!(dh^a + \cdot)$$

and Φ_L^a is equivariant with respect to the translations along the S^1 -direction under $u_3 = x_3$. Here, $\pi : U_L^a \rightarrow L$ is the restriction of the projection map $\pi : T^*L \rightarrow L$.

Proof. The decomposition-by-Lagrangian-subbundles $T_*Y'|_{L'^a} = T_*L'^a \oplus J \cdot T_*L'^a$ and the fact that $T_*L'^a$ is a trivial bundle imply that a Lagrangian distribution along and transverse to L'^a in Y' is the same as a section of the trivial bundle $Mor^{sym}(J \cdot T_*L'^a, T_*L'^a)$ of linear maps from $J \cdot T_pL'^a$ to $T_pL'^a$, $p \in L'^a$, that are represented by symmetric matrices under the canonical isomorphism $T_*L'^a \xrightarrow{\sim} J \cdot T_*L'^a$ by J and a fixed trivialization of $J \cdot T_*L'^a \simeq L'^a \times \mathbb{R}^3$. Under the identification $T^*L' \simeq Y'$, the fibers of $(T^*L')|_{\{|\hat{z}_1| \geq R'_0\}}$ give a transverse Lagrangian foliation along L'^a , and, hence, a smooth map $L'^a \cap \{|\hat{z}_1| \geq R'_0\} \rightarrow \mathbb{R}^3$. It can always be extended to a smooth map $L'^a \rightarrow \mathbb{R}^3$ and, hence, a transverse Lagrangian distribution along L'^a . The linear structure on Y' turns further a transverse Lagrangian distribution along L'^a into a transverse Lagrangian foliation in a neighborhood of L'^a in Y' .

In our situation, all these constructions can be made invariant under the S^1 -translations on Y' in the u_3 -coordinate. The proposition now follows from the proof of Theorem 2.3.2. \square

Definition 2.4.2. [admissible Lagrangian neighborhood for L'^a under ψ^a]. A Lagrangian neighborhood $\Phi_L^a : U_L^a \rightarrow Y'$ for L'^a as in Proposition 2.4.1 for some $R'_0 > 0$ is called an *admissible Lagrangian neighborhood* for L'^a under ψ^a .

Notation 2.4.3. [partial scaling]. Recall the map $t \cdot \psi^a := \psi^{at^{m-1}}$ from Notation 2.2.3. We now extend this to denote

$$\begin{aligned} t \cdot \Phi_L^a &:= \Phi_L^{a,t} : & U_L^{a,t} &\longrightarrow & Y' \\ (x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) &\longmapsto & t \cdot \Phi_L^a(t^{-1/m} \cdot (x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3})) \end{aligned}$$

for a given admissible Lagrangian neighborhood $\Phi_L^a : U_L^a \rightarrow Y'$ for L'^a under ψ^a . Here, $t^{-1/m}$ is the partial scaling on T^*L , defined by

$$(x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) \longmapsto (t^{-1/m}x_1, t^{-1/m}x_2, x_3, t^{-1/m}p_{x_1}, t^{-1/m}p_{x_2}, p_{x_3}),$$

and $U_L^{a,t} = t^{1/m} \cdot U_L^a \subset (T^*L, t^{2(m-1)/m}(dx_1 \wedge dp_{x_1} + dx_2 \wedge dp_{x_2}) + dx_3 \wedge dp_{x_3})$. By construction, $t \cdot \Phi_L^a$ is a symplectic embedding that gives an admissible Lagrangian neighborhood for $t \cdot L'^a$ under $t \cdot \psi^a$. It has the image $t \cdot Im \Psi_L^a$.

Proposition 2.4.4. [radius of fibers of $U_L^{a,t}/L$]. Let $g^{a,t} := (t \cdot \psi^a)^* g'$ be the pull-back metric on L , $0 < t < 1$, and $|\cdot|_{g^{a,t}}$ be the norm on fibers of T^*L via $(g^{a,t})^{-1}$ with respect to the dual basis. Note that for $A'_1 > 0$ small enough, U_L^a contains the neighborhood $\{\alpha \in T^*L : |\alpha|_{g^a} < A'_1\}$ of the zero-section in T^*L , where $g^a := g^{a,1}$. Then, $U_L^{a,t}$ contains the neighborhood $\{\alpha \in T^*L : |\alpha|_{g^{a,t}} < A'_1 t\}$ of the zero-section in T^*L .

Proof. Note that $U_L^{a,t} = t^{1/m} \cdot U_L^a$ as submanifolds in T^*L and that $g^{a,t} \geq t^2 g^a \cdot (t^{-1/m} \cdot)_*$ since $t \cdot \psi^a = t \cdot (\psi^a(t^{-1/m} \cdot))$ and the scaling is only partial. The proposition follows. \square

Continuing the situation in Proposition 2.4.4. Recall from Definition 1.2 the pull-back connection $\hat{\nabla}^{a,t}$ on $U_L^{a,t}$ that is constructed solely by $g^{a,t}$.

Proposition 2.4.5. [compatibility of pull-back connection under partial scaling]. The partial scaling $t^{1/m} \cdot$ on T^*L takes $(U_L^a, \hat{\nabla}^a)$ to $(U_L^{a,t}, \hat{\nabla}^{a,t})$, where $\hat{\nabla}^a := \hat{\nabla}^{a,1}$.

Proof. Consider the associated horizontal distribution $\hat{\nabla}'^{a,t}$ of $\hat{\nabla}^{a,t}$ on the fibered (codimension-0) submanifold $U_{L',at^{m-1}} := \Phi_L^{a,t}(U_L^{a,t}) \subset Y'$ and set $\hat{\nabla}'^{a,t} := \hat{\nabla}'^{a,1}$. Then, the proposition is equivalent to the statement that the partial scaling $t \cdot$ on L' takes $(U_{L',a}, \hat{\nabla}'^{a,t})$ to $(U_{L',at^{m-1}}, \hat{\nabla}'^{a,t})$. Which follows by construction. \square

Continuing the discussion. Recall the standard holomorphic 3-form Ω' on Y' and let $\beta^{a,t} := (\Phi_L^{a,t})^*(\text{Im} \Omega')$.

Proposition 2.4.6. [bound for $\|(\hat{\nabla}^{a,t})^k \beta^{a,t}\|_{C^0}$, $k = 0, 1, 2, 3$]. Assume that $t \in (0, 1]$. Then there exists a constant $A'_4 > 0$ such that $\|(\hat{\nabla}^{a,t})^k \beta^{a,t}\|_{C^0} \leq A'_4 t^{-k}$ for $k = 0, 1, 2, 3$.

Proof. Let $g'^{a,t}$ be the metric on L',at^{m-1} induced by g' on Y' and $\beta'^{a,t} := (\text{Im} \Omega')|_{U_{L',at^{m-1}}}$. As $g^{a,t}$ on L and $\hat{\nabla}^{a,t}, \beta^{a,t}$ on $U_L^{a,t}$ come from the pull-back of $g'^{a,t}$, $\hat{\nabla}'^{a,t}$, and $\beta'^{a,t}$ via $\Phi_L^{a,t}$, we will directly prove the corresponding inequalities on the Y' -side in three steps.

(a) An explicit expression for $\hat{\nabla}'^{a,\diamond}$ in $T^*(T^*L'^{a,\diamond})$. Let $\hat{z}_1 = \hat{r}_1 e^{\sqrt{-1}\hat{\theta}_1}$ and $\hat{z}_2 = \hat{r}_2 e^{\sqrt{-1}\hat{\theta}_2}$. Then

$$L'^{a,\diamond} = \{(\hat{r}_1, \hat{\theta}_1, u_3, \hat{r}_2, \hat{\theta}_2, 0) : \hat{r}_2 = a^{1/m} \hat{r}_1^{1/m}, \hat{\theta}_2 = \hat{\theta}_1/m, \hat{r}_1 > 0\}$$

and $(\hat{r}_1, \hat{\theta}_1, u_3) \in \mathbb{R}^+ \times (\mathbb{R}/(2\pi m)) \times (\mathbb{R}/l)$ serves as a global coordinate chart on $L'^{a,\diamond}$. Let $(\hat{r}_1, \hat{\theta}_1, u_3, s_{\hat{r}_1}, s_{\hat{\theta}_1}, s_{u_3})$ be the induced coordinates on $T_* L'^{a,\diamond}$ through the trivialization of $T_* L'^{a,\diamond}$ by the coordinate frame $(\partial_{\hat{r}_1}, \partial_{\hat{\theta}_1}, \partial_{u_3})$ on $L'^{a,\diamond}$ and $(\hat{r}_1, \hat{\theta}_1, u_3, p_{\hat{r}_1}, p_{\hat{\theta}_1}, p_{u_3})$ be the induced coordinates on $T^* L'^{a,\diamond}$ through the trivialization of $T^* L'^{a,\diamond}$ by the dual coframe $(d\hat{r}_1, d\hat{\theta}_1, du_3)$ on $L'^{a,\diamond}$. In terms of these coordinates,

$$\begin{aligned} g'^{a,\diamond} &:= g'|_{L'^{a,\diamond}} \\ &= \left(1 + m^{-2} a^{2/m} \hat{r}_1^{2(1-m)/m}\right) d\hat{r}_1^2 + \hat{r}_1^2 \left(1 + m^{-2} a^{2/m} \hat{r}_1^{2(1-m)/m}\right) d\hat{\theta}_1^2 + du_3^2 \\ &=: A(\hat{r}_1) d\hat{r}_1^2 + B(\hat{r}_1) d\hat{\theta}_1^2 + du_3^2. \end{aligned}$$

This is a product of a 2-dimensional conformally flat metric with a circle. The Levi-Civita connection $\nabla'^{a,\diamond}$ from $g'^{a,\diamond}$ defines a horizontal distribution ${}^*H'^{a,\diamond}$ in $T^*L'^{a,\diamond}$, given by the kernel of the following \mathbb{R}^3 -valued 1-form on $T^*L'^{a,\diamond}$:

$$\begin{pmatrix} dp_{\hat{r}_1} \\ dp_{\hat{\theta}_1} \\ dp_{u_3} \end{pmatrix} - \begin{pmatrix} \omega_{\hat{r}_1}^{\hat{r}_1} & \omega_{\hat{r}_1}^{\hat{\theta}_1} & 0 \\ \omega_{\hat{\theta}_1}^{\hat{r}_1} & \omega_{\hat{\theta}_1}^{\hat{\theta}_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_{\hat{r}_1} \\ p_{\hat{\theta}_1} \\ p_{u_3} \end{pmatrix}.$$

and a horizontal distribution $*H'^{a,\diamond}$ in $T_*L'^{a,\diamond}$, given by the kernel of the following \mathbb{R}^3 -valued 1-form on $T_*L'^{a,\diamond}$:

$$\begin{pmatrix} ds_{\hat{r}_1} \\ ds_{\hat{\theta}_1} \\ ds_{u_3} \end{pmatrix} + \begin{pmatrix} \omega_{\hat{r}_1}^{\hat{r}_1} & \omega_{\hat{\theta}_1}^{\hat{r}_1} & 0 \\ \omega_{\hat{r}_1}^{\hat{\theta}_1} & \omega_{\hat{\theta}_1}^{\hat{\theta}_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s_{\hat{r}_1} \\ s_{\hat{\theta}_1} \\ s_{u_3} \end{pmatrix}.$$

Here, the 1-forms $\omega_{\bullet}^{\bullet}$ on $L'^{a,\diamond}$ are given by

$$\nabla'^{a,\diamond} \partial_{\hat{r}_1} = \omega_{\hat{r}_1}^{\hat{r}_1} \partial_{\hat{r}_1} + \omega_{\hat{r}_1}^{\hat{\theta}_1} \partial_{\hat{\theta}_1} \quad \text{and} \quad \nabla'^{a,\diamond} \partial_{\hat{\theta}_1} = \omega_{\hat{\theta}_1}^{\hat{r}_1} \partial_{\hat{r}_1} + \omega_{\hat{\theta}_1}^{\hat{\theta}_1} \partial_{\hat{\theta}_1};$$

explicitly,

$$\begin{aligned} \omega_{\hat{r}_1}^{\hat{r}_1} &= \Gamma_{\hat{r}_1 \hat{r}_1}^{\hat{r}_1} d\hat{r}_1 + \Gamma_{\hat{\theta}_1 \hat{r}_1}^{\hat{r}_1} d\hat{\theta}_1 = \frac{1}{2} A(\hat{r}_1)^{-1} \frac{d}{d\hat{r}_1} A(\hat{r}_1) d\hat{r}_1, \\ \omega_{\hat{r}_1}^{\hat{\theta}_1} &= \Gamma_{\hat{r}_1 \hat{r}_1}^{\hat{\theta}_1} d\hat{r}_1 + \Gamma_{\hat{\theta}_1 \hat{r}_1}^{\hat{\theta}_1} d\hat{\theta}_1 = \frac{1}{2} B(\hat{r}_1)^{-1} \frac{d}{d\hat{r}_1} B(\hat{r}_1) d\hat{\theta}_1, \\ \omega_{\hat{\theta}_1}^{\hat{r}_1} &= \Gamma_{\hat{r}_1 \hat{\theta}_1}^{\hat{r}_1} d\hat{r}_1 + \Gamma_{\hat{\theta}_1 \hat{\theta}_1}^{\hat{r}_1} d\hat{\theta}_1 = -\frac{1}{2} A(\hat{r}_1)^{-1} \frac{d}{d\hat{r}_1} B(\hat{r}_1) d\hat{\theta}_1, \\ \omega_{\hat{\theta}_1}^{\hat{\theta}_1} &= \Gamma_{\hat{r}_1 \hat{\theta}_1}^{\hat{\theta}_1} d\hat{r}_1 + \Gamma_{\hat{\theta}_1 \hat{\theta}_1}^{\hat{\theta}_1} d\hat{\theta}_1 = \frac{1}{2} B(\hat{r}_1)^{-1} \frac{d}{d\hat{r}_1} B(\hat{r}_1) d\hat{r}_1. \end{aligned}$$

The coordinate frame $(\partial_{\hat{r}_1}, \partial_{\hat{\theta}_1}, \partial_{u_3}, \partial_{p_{\hat{r}_1}}, \partial_{p_{\hat{\theta}_1}}, \partial_{p_{u_3}})$ on $T^*L'^{a,\diamond}$ specifies a trivialization of the bundle $T_*(T^*L'^{a,\diamond})$ over $T^*L'^{a,\diamond}$ and hence coordinates

$$(\hat{r}_1, \hat{\theta}_1, u_3, p_{\hat{r}_1}, p_{\hat{\theta}_1}, p_{u_3}, \xi_{\hat{r}_1}, \xi_{\hat{\theta}_1}, \xi_{u_3}, \xi_{p_{\hat{r}_1}}, \xi_{p_{\hat{\theta}_1}}, \xi_{p_{u_3}})$$

thereupon. In terms of this, the horizontal distribution in $T_*(T^*L'^{a,\diamond})$ that defines the connection $\hat{\nabla}'^{a,\diamond}$ is given by the kernel of the following \mathbb{R}^6 -valued 1-form on $T_*(T^*L'^{a,\diamond})$:

$$\begin{pmatrix} d\xi_{\hat{r}_1} \\ d\xi_{\hat{\theta}_1} \\ d\xi_{u_3} \\ d\xi_{p_{\hat{r}_1}} \\ d\xi_{p_{\hat{\theta}_1}} \\ d\xi_{p_{u_3}} \end{pmatrix} + \begin{pmatrix} \omega_{\hat{r}_1}^{\hat{r}_1} & \omega_{\hat{\theta}_1}^{\hat{r}_1} & 0 & 0 & 0 & 0 \\ \omega_{\hat{r}_1}^{\hat{\theta}_1} & \omega_{\hat{\theta}_1}^{\hat{\theta}_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_{\hat{r}_1} \\ \xi_{\hat{\theta}_1} \\ \xi_{u_3} \\ \xi_{p_{\hat{r}_1}} \\ \xi_{p_{\hat{\theta}_1}} \\ \xi_{p_{u_3}} \end{pmatrix} =: d\vec{\xi} + \vec{\omega} \vec{\xi}.$$

It follows that in terms of the coordinates $(\hat{r}_1, \hat{\theta}_1, u_3, p_{\hat{r}_1}, p_{\hat{\theta}_1}, p_{u_3})$ on $T^*L'^{a,\diamond}$,

$$\hat{\nabla}'^{a,\diamond} s = ds - {}^t \vec{\omega} s$$

for s a section of the bundle $T^*(T^*L'^{a,\diamond})$ over $T^*L'^{a,\diamond}$ that is trivialized by the coframe $(d\hat{r}_1, d\hat{\theta}_1, du_3, dp_{\hat{r}_1}, dp_{\hat{\theta}_1}, dp_{u_3})$ on $T^*L'^{a,\diamond}$. Here, ${}^t \vec{\omega}$ is the transpose of the matrix-valued 1-form $\vec{\omega}$. One can convert the above expression for $\hat{\nabla}'^{a,\diamond}$ to an expression in terms of the complex-real canonical coordinates $(\hat{z}_1, p_{\hat{z}_1}) := (u_1 + \sqrt{-1}u_2, u_3, p_{u_1} - \sqrt{-1}p_{u_2}, p_{u_3})$ as

$$\begin{aligned} \hat{\nabla}'^{a,\diamond}_{\partial_{\hat{z}_1}} d\hat{z}_1 &= -\frac{m^{-3}(1-m) a^{2/m} \hat{z}_1^{(1-2m)/m} \bar{\hat{z}}_1^{(1-m)/m}}{1 + m^{-2} a^{2/m} \hat{z}_1^{(1-m)/m} \bar{\hat{z}}_1^{(1-m)/m}} d\hat{z}_1, \\ \hat{\nabla}'^{a,\diamond}_{\partial_{\bar{\hat{z}}_1}} d\bar{\hat{z}}_1 &= -\frac{m^{-3}(1-m) a^{2/m} \hat{z}_1^{(1-m)/m} \bar{\hat{z}}_1^{(1-2m)/m}}{1 + m^{-2} a^{2/m} \hat{z}_1^{(1-m)/m} \bar{\hat{z}}_1^{(1-m)/m}} d\bar{\hat{z}}_1, \\ \hat{\nabla}'^{a,\diamond}_{\partial_{\hat{z}_1}} d\hat{z}_1 &= \hat{\nabla}'^{a,\diamond}_{\partial_{\bar{\hat{z}}_1}} d\bar{\hat{z}}_1 = \text{all other } \hat{\nabla}'^{a,\diamond}_{\bullet} = 0. \end{aligned}$$

(b) A uniform bound for $\|(\hat{\nabla}'^a)^k \beta'^a\|_{C^0}$, $k = 0, 1, 2, 3$, for $U_{L',a}$. Recall first some basic variations to express β' on Y' :

$$\begin{aligned}\beta' &:= \operatorname{Im}(dz_1 \wedge dz_2 \wedge dz_3) = \operatorname{Re}(dz_1 \wedge dz_2) \wedge dv_3 + \operatorname{Im}(dz_1 \wedge dz_2) \wedge du_3 \\ &= \frac{\sqrt{-1}}{2}(d\hat{z}_1 \wedge d\bar{\hat{z}}_1 + d\hat{z}_2 \wedge d\bar{\hat{z}}_2) \wedge dv_3 - \operatorname{Im}(d\hat{z}_1 \wedge d\hat{z}_2) \wedge du_3.\end{aligned}$$

To re-express β' in terms of $T^*L'^{a,\diamond}$ in the region $\{|\hat{z}_1| \geq R'_0\}$, consider the smooth map $Y'^{\diamond} \rightarrow Y'^{\diamond}$, $(\hat{z}_1, u_3, \hat{z}_2, v_3) \mapsto (\hat{z}_1, u_3, \hat{z}_2 + a^{1/m} \hat{z}_1^{1/m}, v_3)$. Then, β' is pulled back to

$$\begin{aligned}\beta'|_{T^*L'^{a,\diamond}_{\{|\hat{z}_1| \geq R'_0\}}} &= \frac{\sqrt{-1}}{2} \left((1 + m^{-2} a^{2/m} \hat{z}_1^{(1-m)/m} \bar{\hat{z}}_1^{(1-m)/m}) d\hat{z}_1 \wedge d\bar{\hat{z}}_1 + dp_{\hat{z}_1} \wedge d\bar{p}_{\hat{z}_1} \right) \wedge dp_{u_3} \\ &\quad + \frac{\sqrt{-1}}{2} m^{-1} a^{1/m} \left(\hat{z}_1^{(1-m)/m} d\hat{z}_1 \wedge d\bar{p}_{\hat{z}_1} - \bar{\hat{z}}_1^{(1-m)/m} d\bar{\hat{z}}_1 \wedge dp_{\hat{z}_1} \right) \wedge dp_{u_3} \\ &\quad - \operatorname{Im}(d\hat{z}_1 \wedge dp_{\hat{z}_1}) \wedge du_3\end{aligned}$$

with the coordinates on $T^*L'^{a,\diamond}$ given by $(u_1 + \sqrt{-1}u_2, u_3, p_{u_1} - \sqrt{-1}p_{u_2}, p_{u_3}) =: (\hat{z}_1, u_3, p_{\hat{z}_1}, v_3)$. After a further change of coordinates $(\hat{z}_1, u_3, p_{\hat{z}_1}, v_3) = (\hat{r}_1 e^{\sqrt{-1}\hat{\theta}_1}, u_3, e^{-\sqrt{-1}\hat{\theta}_1}(p_{\hat{r}_1} - \sqrt{-1}\frac{p_{\hat{\theta}_1}}{\hat{r}_1}), p_{u_3})$, it can be expressed also as

$$\begin{aligned}\beta'|_{T^*L'^{a,\diamond}_{\{\hat{r}_1 \geq R'_0\}}} &= \left(\left[(1 + m^{-2} a^{2/m} \hat{r}_1^{2(1-m)/m}) \hat{r}_1 - \hat{r}_1^{-3} p_{\hat{\theta}_1}^2 \right] d\hat{r}_1 \wedge d\hat{\theta}_1 \right. \\ &\quad \left. - \hat{r}_1^{-2} p_{\hat{\theta}_1} d\hat{r}_1 \wedge dp_{\hat{r}_1} + p_{\hat{r}_1} d\hat{\theta}_1 \wedge dp_{\hat{r}_1} + \hat{r}_1^{-2} p_{\hat{\theta}_1} d\hat{\theta}_1 \wedge dp_{\hat{\theta}_1} - \hat{r}_1^{-1} dp_{\hat{r}_1} \wedge dp_{\hat{\theta}_1} \right) \wedge dp_{u_3} \\ &\quad - m^{-1} a^{1/m} \hat{r}_1^{(1-m)/m} \cos\left(\frac{(1+m)\hat{\theta}_1}{m}\right) \left(p_{\hat{r}_1} d\hat{r}_1 \wedge d\hat{\theta}_1 + \hat{r}_1^{-1} d\hat{r}_1 \wedge dp_{\hat{\theta}_1} + \hat{r}_1 d\hat{\theta}_1 \wedge dp_{\hat{r}_1} \right) \wedge dp_{u_3} \\ &\quad + m^{-1} a^{1/m} \hat{r}_1^{(1-m)/m} \sin\left(\frac{(1+m)\hat{\theta}_1}{m}\right) \left(2\hat{r}_1^{-1} p_{\hat{\theta}_1} d\hat{r}_1 \wedge d\hat{\theta}_1 - d\hat{r}_1 \wedge dp_{\hat{r}_1} + d\hat{\theta}_1 \wedge dp_{\hat{\theta}_1} \right) \wedge dp_{u_3} \\ &\quad + \left(p_{\hat{r}_1} d\hat{r}_1 \wedge d\hat{\theta}_1 + \hat{r}_1^{-1} d\hat{r}_1 \wedge dp_{\hat{\theta}_1} - \hat{r}_1 d\hat{\theta}_1 \wedge dp_{\hat{r}_1} \right) \wedge du_3.\end{aligned}$$

To show that $\|(\hat{\nabla}'^a)^k \beta'^a\|_{C^0}$, $k = 0, 1, 2, 3$, is uniformly bounded on $U_{L',a}$, one only needs to show that $\|(\hat{\nabla}'^a)^k \beta'^a\|_{C^0}$, $k = 0, 1, 2, 3$, is uniformly bounded on $U_{L',a} \cap \{|\hat{z}_1| \geq R'_0\}$.

First, recall Definition 1.2 that the horizontal distribution $\hat{\nabla}'^a$ on $U_{L',a}$ is used to define a metric \hat{h}'^a on $U_{L',a}$. From the explicit expression in Part (a) in coordinates $(\hat{z}_1, u_3, p_{\hat{z}_1}, p_{u_3})$, $\hat{\nabla}'^a = O(|\hat{z}_1|^{(2-3m)/m})$ and, hence, $\rightarrow 0$ uniformly on $U_{L',a}$ as $|\hat{z}_1| \rightarrow \infty$. Thus, the $\|\cdot\|_{C^0}$ -norm, with respect to \hat{h}'^a , of tensor product of elements in $\{d\hat{z}_1, d\bar{\hat{z}}_1, du_3, dp_{\hat{z}_1}, d\bar{p}_{\hat{z}_1}, dp_{u_3}\}$ are all uniformly bounded on $U_{L',a}$. It remains to analyze the large- $|\hat{z}_1|$ behavior of the coefficients of $(\hat{\nabla}'^a)^k \beta'^a$ in terms of the basis from these tensor products. Re-write

$$\begin{aligned}\beta'|_{T^*L'^{a,\diamond}_{\{|\hat{z}_1| \geq R'_0\}}} &= \frac{\sqrt{-1}}{2} \left((1 + O(|\hat{z}_1|^{2(1-m)/m})) d\hat{z}_1 \wedge d\bar{\hat{z}}_1 + dp_{\hat{z}_1} \wedge d\bar{p}_{\hat{z}_1} \right) \wedge dp_{u_3} \\ &\quad + \left(O(|\hat{z}_1|^{(1-m)/m}) d\hat{z}_1 \wedge d\bar{p}_{\hat{z}_1} - O(|\bar{\hat{z}}_1|^{(1-m)/m}) d\bar{\hat{z}}_1 \wedge dp_{\hat{z}_1} \right) \wedge dp_{u_3} \\ &\quad - \operatorname{Im}(d\hat{z}_1 \wedge dp_{\hat{z}_1}) \wedge du_3 \\ &\quad \text{as } |\hat{z}_1| \rightarrow \infty.\end{aligned}$$

From the fact that $\hat{\nabla}'^a = O(|\hat{z}_1|^{(2-3m)/m})$ as $|\hat{z}_1| \rightarrow \infty$ and the identities

$$\begin{aligned}(\hat{\nabla}'^a)_{e_{i_1}, e_{i_2}}^2 \beta'^a &= \hat{\nabla}'^a_{e_{i_1}} \hat{\nabla}'^a_{e_{i_2}} \beta'^a - \hat{\nabla}'^a_{\hat{\nabla}'^a_{e_{i_1}} e_{i_2}} \beta'^a, \\ (\hat{\nabla}'^a)_{e_{i_1}, e_{i_2}, e_{i_3}}^3 \beta'^a &= \hat{\nabla}'^a_{e_{i_1}} \hat{\nabla}'^a_{e_{i_2}} \hat{\nabla}'^a_{e_{i_3}} \beta'^a - \hat{\nabla}'^a_{\hat{\nabla}'^a_{e_{i_1}} e_{i_2}} \hat{\nabla}'^a_{e_{i_3}} \beta'^a - \hat{\nabla}'^a_{e_{i_2}} \hat{\nabla}'^a_{\hat{\nabla}'^a_{e_{i_1}} e_{i_3}} \beta'^a \\ &\quad - \hat{\nabla}'^a_{e_{i_1}} \hat{\nabla}'^a_{\hat{\nabla}'^a_{e_{i_2}} e_{i_3}} \beta'^a + \hat{\nabla}'^a_{\hat{\nabla}'^a_{e_{i_1}} e_{i_2}} \hat{\nabla}'^a_{e_{i_3}} \beta'^a + \hat{\nabla}'^a_{\hat{\nabla}'^a_{e_{i_2}} e_{i_3}} \hat{\nabla}'^a_{e_{i_1}} \beta'^a,\end{aligned}$$

one observes that each time a covariant derivative is applied to a term, the large- $|\hat{z}_1|$ behavior of the coefficients of the resulting terms either remains the same or shifts from $O(|\hat{z}_1|^\bullet)$ to $O(|\hat{z}_1|^{\bullet-1})$. It follows that all coefficients of $\beta'|_{T^*L'^{a,\diamond}_{\{|\hat{z}_1| \geq R'_0\}}}$ are uniformly bounded on $U_{L',a}$.

Thus, $\|(\hat{\nabla}'^a)^k \beta'^a\|_{C^0}$, $k = 0, 1, 2, 3$, are uniformly bounded on $U_{L',a}$.

(c) *Bounds for $\|(\hat{\nabla}'^a)^k \beta'^{a,t}\|_{C^0}$, $k = 0, 1, 2, 3$, for $U_{L',at^{m-1}}$.* The scaling argument of Joyce (cf. [Jo3: III, Sec. 6.3]) applies here only better since the scaling involved in our situation is only partial. To proceed, first note that for a diffeomorphism $\tau : M_1 \rightarrow M_2$ on manifolds and k -tensor α and vector fields X_1, \dots, X_k on M_1 , $(\tau_1 \alpha)(\tau_* X_1, \dots, \tau_* X_k) = (\tau^*(\tau^{-1})^* \alpha)(X_1, \dots, X_k) = \tau(X_1, \dots, X_k)$. Let $(e_1, e_2, e_3, e_4, e_5, e_6)$ be the coordinate frame $(\partial_{u_1}, \partial_{u_2}, \partial_{u_3}, \partial_{v_1}, \partial_{v_2}, \partial_{v_3})$ on Y' and $(e^1, e^2, e^3, e^4, e^5, e^6)$ its dual coframe. Since $t_!^{-1} \hat{\nabla}'^{a,t} = \hat{\nabla}'^a$ as horizontal distributions and $t_!^{-1} \beta'^{a,t} = t^2 \beta'^a$ by construction, after passing to $U_{L',at^{m-1}}$ and for $t \in (0, 1]$,

$$\begin{aligned} & ((\hat{\nabla}'^{a,t})_{e_{i_1}, \dots, e_{i_k}}^k \beta'^{a,t})(e_{k+1}, e_{k+2}, e_{k+3}) \quad \text{at } pt \in U_{L',a,t} \\ &= ((t_!^{-1} \hat{\nabla}'^{a,t})_{t_*^{-1} e_{i_1}, \dots, t_*^{-1} e_{i_k}}^k (t_!^{-1} \beta'^{a,t}))(t_*^{-1} e_{k+1}, t_*^{-1} e_{k+2}, t_*^{-1} e_{k+3}) \quad \text{at } t^{-1} \cdot pt \in U_{L',a} \\ &\leq t^{-k} ((\hat{\nabla}'^a)_{e_{i_1}, \dots, e_{i_k}}^k \beta'^a)(e_{k+1}, e_{k+2}, e_{k+3}) \end{aligned}$$

as sections in the contraction $((\otimes_k T^* Y') \otimes \Omega^3(Y')) \otimes (\otimes_{k+3} T_* Y') \rightarrow C^\infty(Y')$ through evaluation, over submanifolds $U_{L',a,t}$, $U_{L',a} \subset Y'$. Here, we used the fact that exactly one of $e_{k+1}, e_{k+2}, e_{k+3}$ must be in $\{e_3, e_6\}$ for the above contraction to be non-zero. Since there exists a constant $C'_1 > 0$ such that $\|e^i\|_{C_{U_{L',a,t}}^0} \leq C'_1 t \|e^i\|_{C_{U_{L',a}}^0}$ for $i = 1, 2, 4, 5$ and $\|e^i\|_{C_{U_{L',a}}^0}$ for $i = 3, 6$, one has

$$\|(\hat{\nabla}'^{a,t})^k \beta'^{a,t}\|_{C_{U_{L',a,t}}^0} \leq C'_2 t^{-k} \|(\hat{\nabla}'^a)^k \beta'^a\|_{C_{U_{L',a}}^0} \leq A'_4 t^{-k},$$

for some constants $C'_2 > 0$ and $A'_4 > 0$, from the uniform bound for $\|(\hat{\nabla}'^a)^k \beta'^a\|_{C^0}$, $k = 0, 1, 2, 3$, for $U_{L',a}$ in Part (b). This proves the proposition. \square

3 Immersed Lagrangian deformations of a simple normalized branched covering of a special Lagrangian 3-sphere in a Calabi-Yau 3-fold and their deviation from Joyce's criteria.

We construct in Sec. 3.1 a natural family of immersed Lagrangian deformations of a simple normalized branched covering of a special Lagrangian 3-sphere in a Calabi-Yau 3-fold and compute in Sec. 3.2 - Sec. 3.4 their deviation from Joyce's criteria.

3.1 Immersed Lagrangian deformations of a branched covering of a special Lagrangian 3-sphere in a Calabi-Yau 3-fold.

Let $Z_0 \simeq S^3 \subset Y$ be a special Lagrangian 3-sphere in a Calabi-Yau 3-fold $Y = (Y, J, \omega, \Omega)$. It follows from [McL] that Z_0 is rigid. The canonical inclusion $J \cdot T_* Z_0 \subset (T_* Y)|_{Z_0}$ gives a distribution on Y along Z_0 that is perpendicular to the canonical inclusion $T_* Z_0 \subset (T_* Y)|_{Z_0}$. Let $\Phi_{Z_0} : U_{Z_0} \rightarrow Y$ be a symplectomorphism from a neighborhood U_{Z_0} of the zero-section of $T^* Z_0$ to a neighborhood of Z_0 in Y such that its restriction to the zero-section is $id_{Z_0} : Z_0 \rightarrow Z_0$ and the embedding $\Phi_{Z_0,*} : T_* U_{Z_0} \rightarrow T_* Y$ sends $T_0(\pi_{Z_0}^{-1}(z))$ to $J \cdot T_z Z_0$ for $z \in Z_0$. Here

$\pi_{Z_0} : U_{Z_0} \rightarrow Z_0$ is the restriction of the bundle map $T^*Z_0 \rightarrow Z_0$ to U_{Z_0} and one endows U_{Z_0} with a Calabi-Yau structure via Φ_{Z_0} .

Simple normalized branched coverings of a sL 3-sphere in a Calabi-Yau 3-fold.

Let

- X be a closed oriented 3-manifold;
- $f : X \rightarrow Z_0$ be a smooth, orientation-preserving, finite, branched covering of Z_0 , with branch locus $\Gamma \subset Z_0$ (downstairs) and $\tilde{\Gamma} \subset X$ (upstairs);
- $\Gamma = \coprod_{i=1}^n \Gamma_i$ and $\tilde{\Gamma} = \coprod_{j=1}^{\tilde{n}_0} \tilde{\Gamma}_j$ be the decomposition of Γ and $\tilde{\Gamma}$ into connected components.

Definition 3.1.1. [simple normalized branched covering]. $f : X \rightarrow Z_0$ is called a *simple normalized* branched covering of Z_0 in Y if it satisfies in addition the following conditions:

- [simple] Each $\Gamma_i, \tilde{\Gamma}_j$ is a smooth 1-submanifold of X isomorphic to a circle S^1 and f maps each $\tilde{\Gamma}_j$ diffeomorphically to some Γ_i ;
- [normalized] When $f : \tilde{\Gamma}_j \rightarrow \Gamma_i$, there exist a tubular neighborhood $\nu_X(\tilde{\Gamma}_j)$ of $\tilde{\Gamma}_j$ in X , a tubular neighborhood $\nu_{Z_0}^j(\Gamma_i)$ of Γ_i in Z_0 , coordinates $(x_1, x_2, x_3) \in \mathbb{R}^2 \times (\mathbb{R}/l_i)$ on $\nu_X(\tilde{\Gamma}_j)$, and coordinates $(u_1, u_2, u_3) \in \mathbb{R}^2 \times (\mathbb{R}/l_i)$ on $\nu_{Z_0}^j(\Gamma_i)$, where l_i is the length of Γ_i in Y , such that the following holds:

- The restriction of $(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3})$ to Γ_i is an orthonormal frame along Γ_i with $\frac{\partial}{\partial u_3}$ tangent to Γ_i .
- The restriction $f : \nu_X(\tilde{\Gamma}_j) \rightarrow \nu_{Z_0}^j(\Gamma_i)$ is given by

$$(u_1, u_2, u_3) = f(x_1, x_2, x_3) = (Re((x_1 + \sqrt{-1}x_2)^{m_j}), Im((x_1 + \sqrt{-1}x_2)^{m_j}), x_3)$$

for some $m_j \in \mathbb{Z}_{\geq 2}$.

m_j is called the *degree/multiplicity/order of f around $\tilde{\Gamma}_j$* .

Remark 3.1.2. [weaker condition]. In the above definition, the requirement that

‘The restriction of $(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_3})$ to Γ_i is an orthonormal frame along Γ_i with $\frac{\partial}{\partial u_3}$ tangent to Γ_i .’

is only for the simplicity of the presentation in this note. It can be replaced by the weaker requirement that

‘The restriction of $\frac{\partial}{\partial u_3}$ to Γ_i is tangent to Γ_i .’

Furthermore, since the construction below is local in nature and the branched covering map has finite degree, the condition that

‘ f maps each $\tilde{\Gamma}_j$ diffeomorphically to some Γ_i .’

can be weakened to the condition

‘ f maps each $\tilde{\Gamma}_j$ to some Γ_i as a smooth finite covering map’.

Auxiliary flat Lagrangian neighborhoods Y'^j , $j = 1, \dots, \tilde{n}_0$, for the branch locus $\Gamma \subset Y$ of f .

The inclusion $\nu_{Z_0}^j(\Gamma_i) \subset Z_0$ induces an inclusion $T^*\nu_{Z_0}^j(\Gamma_i) \subset T^*Z_0$. Let $U_{\nu_{Z_0}^j(\Gamma_i)} := U_{Z_0} \cap T^*\nu_{Z_0}^j(\Gamma_i)$ and $\Phi_{\nu_{Z_0}^j(\Gamma_i)} := \Phi_{Z_0}|_{U_{\nu_{Z_0}^j(\Gamma_i)}}$. Shrinking all these tubular neighborhoods if necessary, one has then a symplectomorphism between tubular neighborhoods

$$\Phi_{\nu_{Z_0}^j(\Gamma_i)} : U_{\nu_{Z_0}^j(\Gamma_i)} \longrightarrow \nu_Y^j(\Gamma_i) \subset Y,$$

whose restriction to the zero-section is the identity map $\nu_{Z_0}^j(\Gamma_i) \rightarrow \nu_{Z_0}^j(\Gamma_i)$.

Recall the coordinates (u_1, u_2, u_3) on $\nu_{Z_0}^j(\Gamma_i)$ and the construction in Sec. 2.2. Identify the canonical coordinates $(u_1, u_2, u_3, p_{u_1}, p_{u_2}, p_{u_3})$ on $U_{\nu_{Z_0}^j(\Gamma_i)} \subset T^*\nu_{Z_0}^j(\Gamma_i)$ here with the coordinates $(u_1, u_2, u_3, v_1, v_2, v_3)$ on Y' there. Then the Calabi-Yau structure (J', ω', Ω') on Y' in Sec. 2.2 induces a Calabi-Yau structure, denoted also by (J', ω', Ω') , on $U_{\nu_{Z_0}^j(\Gamma_i)}$ that is flat. Denote $Y'^j := U_{\nu_{Z_0}^j(\Gamma_i)}$ and $\Phi_{\nu_{Z_0}^j(\Gamma_i)} : U_{\nu_{Z_0}^j(\Gamma_i)} \rightarrow \nu_Y^j(\Gamma_i)$ by

$$\Upsilon^j : (Y'^j, \omega') \longrightarrow (\nu_Y^j(\Gamma_i), \omega) \subset Y.$$

Then the Kähler property of a Calabi-Yau structure and the special Lagrangian property with respect to a calibration imply that

$$\Upsilon^{j*}(J, \Omega)|_{\Gamma_i} = (J', \Omega')|_{\Gamma_i}.$$

In this sense, $(Y'^j, J', \omega', \Omega')$, denoted collectively also by Y'^j , is an *infinitesimal flat approximation* of $(\nu_Y^j(\Gamma_i), J, \omega, \Omega)$ ($=: \nu_Y^j(\Gamma_i)$ collectively) in Y via Υ^j and one can identify Y'^j with a tubular neighborhood of the zero-section of the orthogonal complement $(\frac{\partial}{\partial u_3}|_{\Gamma_i})^\perp$ of the nowhere-zero section $\frac{\partial}{\partial u_3}|_{\Gamma_i}$ in $(T_*Y)|_{\Gamma_i}$, with the induced flat Calabi-Yau structure.

Immersed Lagrangian deformations f^t of f from gluing.

The restriction $f : \nu_X(\tilde{\Gamma}_j) \rightarrow \nu_{Z_0}^j(\Gamma_i) \subset Y$ defines a special Lagrangian map

$$\psi^j : \nu_X(\tilde{\Gamma}_j) \longrightarrow \nu_{Z_0}^j(\Gamma_i) \subset Y'^j$$

such that $f = \Upsilon^j \circ \psi^j$ on $\nu_X(\tilde{\Gamma}_j)$. We'll glue the immersed Lagrangian deformation $\psi^{j, a_j t^{m_j-1}}$ of ψ^j as constructed in Sec. 2.2 to f to give an immersed Lagrangian deformation f^t of f .

The following class of cutoff functions with their first three derivatives bounded in the best possible manner is the basis of our gluing construction and some later estimates:

Lemma 3.1.3. [cutoff function]. *Given $\delta > 0$ and $R_0 > 0$, let $t \in (0, \delta)$ and $0 < b_1^t < b_2^t < R_0$ be constants that depend smoothly on t such that $b_1^t, b_2^t, b_1^t/b_2^t \rightarrow 0$ when $t \rightarrow 0$. Then, there exist smooth functions $\chi^t : (0, R_0) \rightarrow [0, 1]$ that depend smoothly on t as well and a constant $C_0 > 0$, independent of t , such that the following hold:*

- $\chi^t : (0, b_1^t] \rightarrow \{1\}$.
- $\chi^t : [b_2^t, R_0) \rightarrow \{0\}$.
- For t small enough, $\left| \frac{d^k}{dr^k} \chi^t(r) \right| \leq C_0 \cdot (b_2^t)^{-k}$, for $k = 1, 2, 3$.

Proof. For t small enough, one may assume that $0 < b_1^t < \frac{1}{8} \cdot b_2^t$. Consider the following piecewise linear continuous function

$$\hat{\chi}^t(r) = \begin{cases} 0 & \text{for } 0 < r < b_1^t, \\ \frac{a_1}{r_1 - b_1^t} (r - b_1^t) & \text{for } b_1^t \leq r < r_1, \\ \frac{-a_1}{r_2 - r_1} (r - r_1) + a_1 & \text{for } r_1 \leq r < r_2, \\ \frac{a_2}{r_3 - r_2} (r - r_2) & \text{for } r_2 \leq r < r_3, \\ \frac{-a_2}{r_4 - r_3} (r - r_3) + a_2 & \text{for } r_3 \leq r < r_4, \\ \frac{a_3}{r_5 - r_4} (r - r_4) & \text{for } r_4 \leq r < r_5, \\ \frac{-a_3}{b_2^t - r_5} (r - r_5) + a_3 & \text{for } r_5 \leq r < b_2^t, \\ 0 & \text{for } b_2^t \leq r < R_0. \end{cases}$$

which depends on the parameters $a_1, a_3 < 0$; $a_2 > 0$; $b_1^t < r_1 < r_2 < r_3 < r_4 < r_5 < b_2^t$ with r_1 (resp. r_2, r_3, r_4, r_5) in a small neighborhood of $\frac{1}{8} b_2^t$ (resp. $\frac{1}{4} b_2^t, \frac{1}{2} b_2^t, \frac{3}{4} b_2^t, \frac{7}{8} b_2^t$). By an appropriate adjustment of these parameters and a smoothing $\hat{\chi}^{t,\sim}$ of $\hat{\chi}^t$ in the C^∞ -topology, one can choose χ^t as required by taking

$$\chi^t(r) = 1 + \int_0^r \int_0^{r'} \int_0^{r''} \hat{\chi}^{t,\sim}(r''') dr''' dr'' dr'.$$

□

We'll denote $d\chi^t/dr$ and $d^2\chi^t/dr^2$ also by $\dot{\chi}^t$ and $\ddot{\chi}^t$ respectively.

Definition 3.1.4. [immersed Lagrangian deformations f^t of f from gluing]. Let

$$\begin{aligned} X^\diamond &:= X - \tilde{\Gamma}, & Z_0^\diamond &:= Z_0 - \Gamma, \\ \nu_X(\tilde{\Gamma}_j)^\diamond &:= \nu_X(\tilde{\Gamma}_j) - \tilde{\Gamma}_j = \nu_X(\tilde{\Gamma}_j) \cap X^\diamond, & \nu_{Z_0}^j(\Gamma_i)^\diamond &:= \nu_{Z_0}^j(\Gamma_i)^\diamond = \nu_{Z_0}^j(\Gamma_i) \cap Z_0^\diamond. \end{aligned}$$

The restriction $f^\diamond : X^\diamond \rightarrow Z_0^\diamond$ of f to X^\diamond is a covering map and, hence, induces a covering map

$$f_!^\diamond : T^*X^\diamond \longrightarrow T^*Z_0^\diamond$$

that is a local symplectomorphism. Recall Sec. 2.2 and the notations therein. For a pair (j, i) with $f(\tilde{\Gamma}_j) = \Gamma_i$, let $R_0 > 0$ and $a_j > 0$ be small enough so that

- the solid torus $\{|\hat{z}_1| \leq R_0\}$ around Γ_i in Z_0 is contained in $\nu_{Z_0}^j(\Gamma_i)$ and, for notational convenience, we shrink $\nu_{Z_0}^j(\Gamma_i)$ so that they are the same from now on,
- the solid torus $\{|x_1 + \sqrt{-1}x_2| \leq R_0^{1/m_j}\}$ around $\tilde{\Gamma}_j$ in X is contained in $\nu_X(\tilde{\Gamma}_j)$ and we now shrink $\nu_X(\tilde{\Gamma}_j)$ so that they are the same from now on,
- the smooth embedded special Lagrangian submanifold with boundary

$$L_{R_0}'^{a_j} := \{|\hat{z}_1| \leq R_0, a_j \hat{z}_1 - \hat{z}_2^{m_j} = 0, v_3 = 0\} \subset Y'$$

is contained in $Y'^{j,j}$.

Recall the graph $\Gamma(\alpha^j)$ of the associated exact 1-form

$$\alpha^j = dh^j := \text{Re} \left(\frac{m_j}{m_j + 1} a_j^{1/m_j} d((x_1 + \sqrt{-1}x_2)^{m_j+1}) \right)$$

on $\nu_X(\tilde{\Gamma}_j)$ whose restriction over $\nu_X(\tilde{\Gamma}_j)^\diamond$ is mapped to $L'_{R_0} \cap \nu_{Z_0}^j(\Gamma_i)^\diamond$ under $f_!^\diamond$. (Cf. Lemma 2.2.2.) While $f_!^\diamond$ does not extend to fibers of T^*X over $\tilde{\Gamma}$, it follows from the explicit study in Sec. 2.2 that the restriction of $f_!^\diamond$ on $\Gamma(\alpha^j)|_{\nu_X(\tilde{\Gamma}_j)^\diamond}$ extends to the whole $\Gamma(\alpha^j)$ and defines a smooth Lagrangian embedding

$$\psi^{j,a_j} : \nu_X(\tilde{\Gamma}_j) \hookrightarrow Y'^j$$

with the image $L'_{R_0}{}^{a_j}$ and satisfying $\pi \circ \psi^{j,a_j} = f|_{\nu_X(\tilde{\Gamma}_j)}$, where $\pi : T^*Z_0 \rightarrow Z_0$ is the bundle map.

Let $t \in (0, \delta)$, $0 < b_1^t < b_2^t < R_0$, and $\chi^t : (0, R_0) \rightarrow [0, 1]$ with $r = |(x_1 + \sqrt{-1}x_2)^{m_j}|$ be as in Lemma 3.1.3. Recall the partial scaling $t \cdot L'^{a_j} := L'^{a_j t^{m_j-1}}$ of L'^{a_j} in Y' ; cf. Notation 2.2.3. The associated 1-form on $\nu_X(\tilde{\Gamma}_j)$ is thus $t^{(m_j-1)/m_j} \alpha^j = d(t^{(m_j-1)/m_j} h^j)$. Define

$$h^{j,t} := \chi^t \cdot (t^{(m_j-1)/m_j} \cdot h^j) \quad \text{and} \quad \alpha^{j,t} = dh^{j,t}.$$

Then, since the two graphs $\Gamma(\alpha^{j,t})$, $\Gamma(t^{(m_j-1)/m_j} \cdot \alpha^j) \subset T^*\nu_X(\tilde{\Gamma}_j)$ of 1-forms are identical over $\{|x_1 + \sqrt{-1}x_2|^{m_j} \leq b_1^t\} \subset \nu_X(\tilde{\Gamma}_j)$, the restriction of $f_!^\diamond$ on $\Gamma(\alpha^{j,t})|_{\nu_X(\tilde{\Gamma}_j)^\diamond}$ extends to the whole $\Gamma(\alpha^{j,t})$ as well. Since $\Gamma(t^{(m_j-1)/m_j} \cdot \alpha^j)$ defines also a smooth Lagrangian embedding of $\nu_X(\tilde{\Gamma}_j)$ under the extension of $f_!^\diamond$ and f is a smooth Lagrangian immersion of $\nu_X(\tilde{\Gamma}_j)^\diamond$, $f_!^\diamond$ defines now a smooth Lagrangian immersion: (following the notation of Sec. 2.2)

$$\psi^{j,a_j t^{m_j-1}} : \nu_X(\tilde{\Gamma}_j) \hookrightarrow Y'^j$$

with the image $L'_{R_0}{}^{a_j t^{m_j-1}}$ and satisfying also $\pi \circ \psi^{j,a_j t^{m_j-1}} = f|_{\nu_X(\tilde{\Gamma}_j)}$. After the post-composition with $\Upsilon^j : Y'^j \rightarrow Y$, one has then a smooth immersion

$$\Upsilon^j \circ \psi^{j,a_j t^{m_j-1}} : \nu_X(\tilde{\Gamma}_j) \longrightarrow Y$$

of Lagrangian submanifold with boundary. Since

$$\chi^t|_{[b_2^t, R_0)} \equiv 0 \quad \text{and} \quad (\Upsilon^j \circ \psi^{j,a_j t^{m_j-1}})|_{\{b_2^t \leq |(x_1 + \sqrt{-1}x_2)^{m_j}| < R_0\}} \equiv f|_{\{b_2^t \leq |(x_1 + \sqrt{-1}x_2)^{m_j}| < R_0\}},$$

it follows that

$$\coprod_{j=1}^{\tilde{n}_0} (\Upsilon^j \circ \psi^{j,a_j t^{m_j-1}}) : \coprod_{j=1}^{\tilde{n}_0} \nu_X(\tilde{\Gamma}_j) \longrightarrow Y$$

can be extended by f on $X - \coprod_j \nu_X(\tilde{\Gamma}_j)$ to a smooth Lagrangian immersion

$$f^t : N^t = X \longrightarrow Y.$$

In other words, recall the Lagrangian neighborhood $\Phi_{Z_0} : U_{Z_0} \rightarrow Y$ of $Z_0 \subset Y$ and the projection map $\pi_{Z_0} : U_{Z_0} \rightarrow Z_0$ at the beginning of this subsection. Then, the smooth function

$$\coprod_{j=1}^{\tilde{n}_0} h^{j,t} : \coprod_{j=1}^{\tilde{n}_0} \nu_X(\tilde{\Gamma}_j) \longrightarrow \mathbb{R}$$

extends to a smooth function

$$h^t : X \longrightarrow \mathbb{R}$$

by 0; $f_!^\diamond$ extends to $f_!$ on the graph $\Gamma(dh^t)$ of dh^t in T^*X ; and

$$f^t = \Phi_{Z_0} \circ f_! \circ dh^t,$$

where dh^t is regarded as a section $X \rightarrow T^*X$ of T^*X over X .

By construction, $f^t \rightarrow f =: f^0$, as $t \rightarrow 0$, both in the sense of currents and in the sense of C^∞ -topology on any compact subset of X^\diamond , and $\pi_{Z_0} \circ f^t = f$ for all $t \in (0, \delta)$.

Notation 3.1.5. [pull-back metric]. Let g_Y be the Calabi-Yau metric on Y determined by (J, ω) . Denote by g^t the pull-back metric $(f^t)^*g_Y$ on $N^t = X$.

Remark 3.1.6. [expression in Y'^j]. By construction, the only difference of f^t and $f =: f^0$ lies in $\nu_X(\tilde{\Gamma})$, whose image in Y lies in $\cup_j(\Upsilon^j(Y'^j))$. In terms of the complex-real coordinates $(u_1 + \sqrt{-1}u_2, u_3, v_1 - \sqrt{-1}v_2, v_3) = (\hat{z}_1, u_2, \hat{z}_2, v_3) = (r_1 e^{\sqrt{-1}\theta_1}, u_3, r_2 e^{\sqrt{-1}\theta_2}, v_3)$ on Y'^j and, hence, on $\Upsilon^j(Y'^j)$, the immersed image $f^t(N^t) \cap \Upsilon^j(Y'^j)$ is given by

$$f^t(N^t) \cap \Upsilon^j(Y'^j) = \left\{ (r_1 e^{\sqrt{-1}\theta_1}, u_3, r_2 e^{\sqrt{-1}\theta_2}, 0) \mid \begin{aligned} \cdot r_2 &= a^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right] \\ \cdot m_j \theta_2 &= \theta_1 \pmod{2\pi} \end{aligned} \right\}.$$

Notation 3.1.7. [f^t in three parts]. With the notation in this subsection, let

$$\begin{aligned} P_j^t &:= \{(x_1, x_2, x_3) \in \nu_X(\tilde{\Gamma}_j) : 0 \leq |(x_1 + \sqrt{-1}x_2)^{m_j}| \leq b_1^t\}, \\ Q_j^t &:= \{(x_1, x_2, x_3) \in \nu_X(\tilde{\Gamma}_j) : b_1^t \leq |(x_1 + \sqrt{-1}x_2)^{m_j}| \leq b_2^t\}, \\ K^t &:= X - \coprod_{j=1}^{\tilde{n}_0} (P_j \cup Q_j). \end{aligned}$$

Then, from the gluing construction of f^t ,

$$f^t = \left(\cup_{j=1}^{\tilde{n}_0} (f^t|_{P_j^t} \cup f^t|_{Q_j^t}) \right) \cup f^t|_{K^t}.$$

3.2 Estimating $Im \Omega|_{N^t}$.

We now estimate the Sobolev norms of $Im \Omega|_{N^t}$ in Criterion (i) of Theorem 1.1. The discussion is based upon Taylor's formula and a finite-dimensional nature of the problem.

Taylor's formula.

Let f be an \mathbb{R} -valued function on an open subset $S \subset \mathbb{R}^m$ and $\mathbf{y} := (y_1, \dots, y_m)$ be the coordinates on \mathbb{R}^m . For $\mathbf{a} \in S$ and $\mathbf{t} \in \mathbb{R}^m$, if all l -th order partial derivatives of f exist at \mathbf{a} , then write

$$f^{(l)}(\mathbf{a}; \mathbf{t}) := \sum_{j_l=1}^m \cdots \sum_{j_1=1}^m \frac{\partial^l f}{\partial y_{j_l} \cdots \partial y_{j_1}}(\mathbf{a}) t_{j_1} \cdots t_{j_l}.$$

Theorem 3.2.1. [Taylor's formula with remainder]. (E.g. [Ap: Theorem 12.14].) *Assume that f and all its partial derivatives of order $\leq l$ are differentiable at each point of an open set S in \mathbb{R}^m . If \mathbf{a} and \mathbf{b} are two points of S such that the line segment $\overline{\mathbf{a}, \mathbf{b}}$ in \mathbb{R}^m that connects \mathbf{a} and \mathbf{b} is contained in S , then there is a point $\mathbf{c} \in \overline{\mathbf{a}, \mathbf{b}}$ such that*

$$f(\mathbf{b}) = \sum_{i=0}^l \frac{1}{i!} f^{(i)}(\mathbf{a}; \mathbf{b} - \mathbf{a}) + \frac{1}{(l+1)!} f^{(l+1)}(\mathbf{c}; \mathbf{b} - \mathbf{a}).$$

Oriented-Lagrangian Grassmannian bundles, prolongation of Lagrangian immersions, and calibrations.

Let (Y, J, ω, Ω) be a Calabi-Yau m -fold, denoted collectively by Y , and $Gr^{L^+}(T_*Y)$ be the oriented-Lagrangian Grassmannian bundle over Y , whose fiber over $y \in Y$ is given by the Grassmannian manifold $Gr^{L^+}(T_y Y)$ of oriented Lagrangian subspaces of $T_y Y$. By construction, a Lagrangian immersion $f : X \rightarrow Y$ from an oriented m -manifold X to Y has a unique lifting

$$\begin{array}{ccc} & & Gr^{L^+}(T_*Y) \\ & \nearrow^{Gr^{L^+}f} & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array},$$

defined by $(Gr^{L^+}f)(x) = [f_*(T_x X)] \in Gr^{L^+}(T_{f(x)} Y)$ for $x \in X$, where $f_*(T_x X)$ is equipped with an orientation from that of $T_x X$ via the isomorphism $f_* : T_x X \rightarrow f_*(T_x X)$.

Definition 3.2.2. [prolongation of Lagrangian immersion]. $Gr^{L^+}f : X \rightarrow Gr^{L^+}(T_*Y)$ is called the *prolongation* of the Lagrangian map $f : X \rightarrow Y$ to $Gr^{L^+}(T_*Y)$.

The holomorphic m -form Ω on Y defines a map $e^{\sqrt{-1}\alpha} : Gr^{L^+}(T_*Y) \rightarrow U(1) \subset \mathbb{C}^*$ by $[L] \mapsto \Omega|_L / vol_L$, where vol_L is the volume-form on L induced by the metric on Y . The imaginary part $\varepsilon := \sin \alpha$ of $e^{\sqrt{-1}\alpha}$ defines a smooth function on $Gr^{L^+}(T_*Y)$. This defines in turn a smooth section $d\varepsilon$ of $T^*(Gr^{L^+}(T_*Y))$ and a smooth function $|d\varepsilon|^2 := |(Gr^{L^+}f)^* d\varepsilon|^2$ on X , using the pullback metric tensor on X under f .

Local charts on $Gr^{L^+}(T_*Y)$ and prolongations as 2-jets.

Note that any Lagrangian tangent subspace of Y is tangent to some embedded Lagrangian submanifold of Y . Local charts on $Gr^{L^+}(T_*Y)$ can thus be provided by Lagrangian neighborhoods on Y as follows.⁶ Let $Z \subset Y$ be an oriented embedded Lagrangian submanifold Y , $\tilde{Z} \subset Gr^{L^+}(T_*Y)$ be its prolongation to $Gr^{L^+}(T_*Y)$, and $\Phi_Z : U_Z \rightarrow Y$ be a Lagrangian neighborhood of $Z \subset Y$, where U_Z is a neighborhood of the zero-section of T^*Y . Local coordinates (u_1, \dots, u_m) of a chart U on Z , with the orientation specified by $du_1 \wedge \dots \wedge du_m$ induce local coordinates $(u_1, \dots, u_m, p_{u_1}, \dots, p_{u_m}) =: (\mathbf{u}, \mathbf{p}_u) \in \mathbb{R}_{(1)}^m \times \mathbb{R}_{(2)}^m$ on the associated chart on U_Z with

$$(u_1, \dots, u_m, p_{u_1}, \dots, p_{u_m}) \longleftrightarrow p_{u_1} du_1 + \dots + p_{u_m} du_m \in T^*Z.$$

In terms of this and by [McD-S: Lemma 2.28], a coordinate chart \tilde{U} for a neighborhood of $\tilde{Z} \subset Gr^{L^+}(T_*Y)$ is given by

$$\{((\mathbf{u}, \mathbf{p}_u), A) \mid (\mathbf{u}, \mathbf{p}_u) \in \text{chart on } U_Z \text{ associated to } U; A : \text{symmetric } m \times m\text{-matrix}\}.$$

Here, $((\mathbf{u}, \mathbf{p}_u), A)$ specifies the oriented Lagrangian tangent subspace at $(\mathbf{u}, \mathbf{p}_u)$ given by

$$\Lambda_{((\mathbf{u}, \mathbf{p}_u), A)} := \{(\xi, A\xi) : \xi \in \mathbb{R}_{(1)}^m\} \subset \mathbb{R}_{(1)}^m \times \mathbb{R}_{(2)}^m,$$

where we have identified the tangent space of a point on $\mathbb{R}_{(1)}^m \times \mathbb{R}_{(2)}^m$ canonically with $\mathbb{R}_{(1)}^m \times \mathbb{R}_{(2)}^m$ itself, using the linear structure, and the orientation of $\Lambda_{((\mathbf{u}, \mathbf{p}_u), A)}$ is specified by the orientation on $\mathbb{R}_{(1)}^m$ via the restriction of the projection map $\mathbb{R}_{(1)}^m \times \mathbb{R}_{(2)}^m \rightarrow \mathbb{R}_{(1)}^m$ to $\Lambda_{((\mathbf{u}, \mathbf{p}_u), A)}$.

⁶Such local charts on $Gr^{L^+}(T_*Y)$ are more convenient for our purpose. One can also consider local charts on $Gr^{L^+}(T_*Y)$ induced by Darboux charts on Y .

Now, an oriented embedded Lagrangian submanifold Z' in Y that is C^1 -close to Z can be expressed as the graph of a closed 1-form on Z under Φ_Z . On a small enough chart U on Z , Z' is thus given by

$$\Gamma_{dh} = \left\{ \left(\mathbf{u}, \frac{\partial h}{\partial u_1}(\mathbf{u}), \dots, \frac{\partial h}{\partial u_m}(\mathbf{u}) \right) \middle| \mathbf{u} \in U \right\}$$

for some $h \in C^\infty(U)$. The prolongation \tilde{Z}' of Z' to $Gr^{L^+}(T_*Y)$ is thus locally given by

$$\tilde{\Gamma}_{dh} = \left\{ \left(\mathbf{u}, \frac{\partial h}{\partial u_1}(\mathbf{u}), \dots, \frac{\partial h}{\partial u_m}(\mathbf{u}), \left(\frac{\partial^2 h}{\partial u_i \partial u_j}(\mathbf{u}) \right)_{j,i} \right) \middle| \mathbf{u} \in U \right\}.$$

It follows that the prolongation $Gr^{L^+}f : X \rightarrow Gr^{L^+}(T_*Y)$ of an oriented Lagrangian immersion $f : X \rightarrow Y$ can be expressed locally as the prolongation of a 2-jet.

Estimating $Im \Omega|_{N^t}$.

Definition/Notation 3.2.3. [setup: reference map of prolongation]. Recall Notation 3.1.7: The submanifolds with boundary $P_j^t, Q_j^t, K^t \subset N^t = X$ and the decomposition

$$f^t = \left(\bigcup_{j=1}^{\tilde{n}_0} (f^t|_{P_j^t} \cup f^t|_{Q_j^t}) \right) \bigcup f^t|_{K^t}$$

of Lagrangian immersions $f^t : X \rightarrow Y$. With respect to this decomposition, define

$$\Theta^t := \left(\prod_{j=1}^{\tilde{n}_0} \left(\Theta_{P_j^t}^t \amalg \Theta_{Q_j^t}^t \right) \right) \amalg \Theta_{K^t}^t : \left(\prod_{j=1}^{\tilde{n}_0} \left(P_j^t \amalg Q_j^t \right) \right) \amalg K^t \longrightarrow Gr^{L^+}(T_*Y)$$

by

$$\begin{aligned} \Theta_{P_j^t}^t : P_j^t &\longrightarrow Gr^{L^+}((T_*Y)|_{\Gamma_i}) \subset Gr^{L^+}(T_*Y) \\ (x_1, x_2, x_3) &\longmapsto pr_2^{Gr}([f_*^t(T_{(x_1, x_2, x_3)} P_j^t)]) \in Gr^{L^+}(T_{(0,0,x_3)} Y), \end{aligned}$$

$$\Theta_{Q_j^t}^t = Gr^{L^+}(f^0|_{Q_j^t}), \quad \text{and} \quad \Theta_{K^t}^t = Gr^{L^+}(f^0|_{K^t}).$$

Here,

- $f(\tilde{\Gamma}_j) = \Gamma_i$;
- the symplectic coordinates $(u_1, u_2, u_3, p_{u_1}, p_{u_2}, p_{u_3}) = (u_1, u_2, u_3, v_1, v_2, v_3)$ on $\nu_{Z_0}^j(\Gamma_i)$ induces a trivialization $Gr^{L^+}(T_*\nu_{Z_0}^j(\Gamma_i)) \simeq T_*\nu_{Z_0}^j(\Gamma_i) \times_{\Gamma_i} Gr^{L^+}(T_*Y|_{\Gamma_i})$ via the symplectic linear structure from the coordinates and pr_2^{Gr} is the projection map to the second factor;
- recall that $f^0 := f$.

We'll call Θ^t a (*piecewise-smooth*) *reference map* for the prolongation $Gr^{L^+}f^t : X \rightarrow Gr^{L^+}(T_*Y)$ of f^t .

Proposition 3.2.4. [basic estimate]. *In the situation and notations in Definition 3.2.3, making $\delta > 0$ smaller if necessary and assuming that $b_1^t = t^{c_1}$, $b_2^t = t^{c_2}$ for some $0 < c_2 < c_1$,*

then there exists a constant $C > 0$ such that for all $t \in (0, \delta)$, one has

$$|\varepsilon^t| \leq \begin{cases} C t^{c_1} + C t^{(1-\frac{1}{m_j})+\frac{c_1}{m_j}} & \text{on } P_j^t, \\ C t^{1-\frac{1}{m_j}-2c_2} + C t^{(1-\frac{1}{m_j})(1-c_1)} & \text{on } Q_j^t, \\ 0 & \text{on } K^t; \end{cases}$$

$$|d\varepsilon^t| \leq \begin{cases} C + C t^{(m_j-1)(c_1-1)/m_j} & \text{on } P_j^t, \\ C t^{1-\frac{1}{m_j}-3c_2} + C t^{(1-\frac{1}{m_j})(1-c_1)-c_2} + C t^{(1-\frac{1}{m_j})-c_1(2-\frac{1}{m_j})} & \text{on } Q_j^t, \\ 0 & \text{on } K^t \end{cases}$$

for all $j = 1, \dots, \tilde{n}_0$. Here $|\cdot|$ is computed using the metric g^t on N^t .

Proof. (See Item (a.1) and Item (b.1) in the proof of Proposition 3.2.5, where all the necessary expressions are collected.) Note that $(\Theta^t)^*\varepsilon^t \equiv 0 \equiv (\Theta^t)^*(d\varepsilon)$ on $(\coprod_{j=1}^{\tilde{n}_0}(\Theta_{P_j^t}^t \amalg \Theta_{Q_j^t}^t)) \amalg \Theta_{K^t}^t$. All the estimates can be made with $P_j^t \cup Q_j^t \subset (X, g^t)$ approximated by the flat geometry on L'^j . Recall Remark 3.1.6. The estimate for ε^t follows thus from pointwise Taylor's formula over X for ε on $Gr^{L^+}(T_*Y)$ with the corresponding point in $Im\Theta^t$ as the reference point and the following estimates: For P_j^t , consider $t \cdot L'^j$ with $r_1 \leq b_1^t$. Then,

$$0 \leq r_1 \leq b_1^t = t^{c_1},$$

$$0 \leq r_2 = a_j^{1/m_j} t^{(m_j-1)/m_j} r_1^{1/m_j} \leq O(t^{(m_j-1)/m_j}) (b_1^t)^{1/m_j} = O(t^{(1-\frac{1}{m_j})+\frac{c_1}{m_j}}).$$

For Q_j^t , consider $t \cdot L'^j$ with $b_1^t \leq r_1 \leq b_2^t$: Then,

$$r_2 = a_j^{1/m_j} t^{(m_j-1)/m_j} r_1^{1/m_j} \geq O(t^{(m_j-1)/m_j}) (b_1^t)^{1/m_j} \geq O(t^{(1-\frac{1}{m_j})+\frac{c_1}{m_j}});$$

$$r_2 = a_j^{1/m_j} t^{(m_j-1)/m_j} r_1^{1/m_j} \leq O(t^{(m_j-1)/m_j}) (b_2^t)^{1/m_j} = O(t^{(1-\frac{1}{m_j})+\frac{c_2}{m_j}});$$

$$\begin{aligned} & \left| a_j^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right] \right| \\ & \leq a_j^{1/m_j} t^{(m_j-1)/m_j} \left(\frac{m_j}{m_j+1} \cdot \frac{C_0}{b_2^t} + r_1^{1/m_j} \right) \\ & \leq O(t^{(m_j-1)/m_j}) (b_2^t)^{-1} + O(t^{(m_j-1)/m_j}) (b_2^t)^{1/m_j} \\ & = O(t^{1-\frac{1}{m_j}-c_2}) + O(t^{1-\frac{1}{m_j}+\frac{c_2}{m_j}}) \\ & = O(t^{1-\frac{1}{m_j}-c_2}); \end{aligned}$$

$$\begin{aligned}
& \left| \frac{d}{dr_1} \left(a_j^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right] \right) \right| \\
&= a_j^{1/m_j} t^{(m_j-1)/m_j} \left| \frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \dot{\chi}^t(r_1) r_1^{1/m_j} + \frac{1}{m_j} \chi^t(r_1) r_1^{(1-m_j)/m_j} \right| \\
&\leq a_j^{1/m_j} t^{(m_j-1)/m_j} \left(\frac{m_j}{m_j+1} \cdot \frac{C_0}{(b_2^t)^2} + \frac{C_0}{b_2^t} r_1^{1/m_j} + \frac{1}{m_j} r_1^{(1-m_j)/m_j} \right) \\
&\leq O(t^{(m_j-1)/m_j} (b_2^t)^{-2}) + O(t^{(m_j-1)/m_j} (b_2^t)^{(1-m_j)/m_j}) + O(t^{(m_j-1)/m_j} (b_1^t)^{(1-m_j)/m_j}) \\
&= O(t^{1-\frac{1}{m_j}-2c_2}) + O(t^{(1-\frac{1}{m_j})(1-c_2)}) + O(t^{(1-\frac{1}{m_j})(1-c_1)}) \\
&= O(t^{1-\frac{1}{m_j}-2c_2}) + O(t^{(1-\frac{1}{m_j})(1-c_1)}).
\end{aligned}$$

Here, we have used the fact that r_1^{1/m_j} (resp. $r_1^{(1-m_j)/m_j}$) is an increasing (resp. decreasing) function over $r_1 > 0$ for all j and $(1 - \frac{1}{m_j})(1 - c_2) > (1 - \frac{1}{m_j})(1 - c_1)$. Under the assumption, all the exponents in the t -orders are positive.

Similarly for the estimates for $|d\varepsilon^t|$ with, in addition, the following estimates: For P_j^t ,

$$dr_2/dr_2 = 1,$$

$$\left| \frac{d}{dr_2} \left(a_j^{-1} t^{1-m_j} r_2^{m_j} \right) \right| = m_j a_j^{-1} t^{1-m_j} r_2^{m_j-1} \leq O(t^{(m_j-1)(c_1-1)/m_j}).$$

For Q_j^t ,

$$\begin{aligned}
& \left| \frac{d^2}{dr_1^2} \left(a_j^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right] \right) \right| \\
&= a_j^{1/m_j} t^{(m_j-1)/m_j} \left| \frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \ddot{\chi}^t(r_1) r_1^{1/m_j} \right. \\
&\quad \left. + \frac{2}{m_j} \dot{\chi}^t(r_1) r_1^{(1-m_j)/m_j} + \frac{1-m_j}{m_j^2} \chi^t(r_1) r_1^{(1-2m_j)/m_j} \right| \\
&\leq a_j^{1/m_j} t^{(m_j-1)/m_j} \left(\frac{m_j}{m_j+1} \frac{C_0}{(b_2^t)^3} + \frac{C_0}{(b_2^t)^2} r_1^{1/m_j} \right. \\
&\quad \left. + \frac{2C_0}{m_j b_2^t} r_1^{(1-m_j)/m_j} + \frac{1-m_j}{m_j^2} r_1^{(1-2m_j)/m_j} \right) \\
&\leq O(t^{(m_j-1)/m_j} (b_2^t)^{-3}) + O(t^{(m_j-1)/m_j} (b_2^t)^{(1-2m_j)/m_j}) \\
&\quad + O(t^{(m_j-1)/m_j} (b_2^t)^{-1} (b_1^t)^{(1-m_j)/m_j}) + O(t^{(m_j-1)/m_j} (b_1^t)^{(1-2m_j)/m_j}) \\
&= O(t^{1-\frac{1}{m_j}-3c_2}) + O(t^{(1-\frac{1}{m_j})-c_2(2-\frac{1}{m_j})}) + O(t^{(1-\frac{1}{m_j})(1-c_1)-c_2}) + O(t^{(1-\frac{1}{m_j})-c_1(2-\frac{1}{m_j})}) \\
&= O(t^{1-\frac{1}{m_j}-3c_2}) + O(t^{(1-\frac{1}{m_j})(1-c_1)-c_2}) + O(t^{(1-\frac{1}{m_j})-c_1(2-\frac{1}{m_j})}).
\end{aligned}$$

Here, we have used in addition the fact that $r_1^{(1-2m_j)/m_j}$ is a decreasing function over $r_1 > 0$ for all j , and $(1 - \frac{1}{m_j}) - c_2(2 - \frac{1}{m_j}) > (1 - \frac{1}{m_j}) - c_1(2 - \frac{1}{m_j})$. Under the assumption, all the exponents in the t -orders are positive.

□

Proposition 3.2.5. [Sobolev norm estimate]. *Continuing the situation in Proposition 3.2.4, with an additional assumption that $m_j \notin \{2, 6, 11\}$, then for all $t \in (0, \delta)$, with δ small enough, the norms $\|\varepsilon^t\|_{L^{6/5}}$, $\|\varepsilon^t\|_{C^0}$, $\|d\varepsilon^t\|_{L^6}$, and $\|\varepsilon^t\|_{L^1}$ are bounded above by t -powers with exponents a linear function in c_1 and c_2 with coefficients fractional functions in m_j , $j = 1, \dots, \tilde{n}_0$.*

Here the norms $\|\cdot\|_\bullet$ are computed using g^t on N^t .

Exact expressions for these exponents are given in the proof.

Proof. The calculation is similar to that in the proof of Proposition 3.2.4, with the same reference map Θ^t for Taylor expansion and an additional ingredient from the t -dependent volume-forms on X from the pull-back metric g^t . As f^t , $t \in (0, \delta)$, are immersions, we will perform the computation using data and coordinates on Y .

(a) On P_j^t .

(a.1) *Basic data for estimates on P_j^t .* Consider

$$f^t(P_j^t) = \left\{ (r_1 e^{\sqrt{-1}\theta_1}, u_3, r_2 e^{\sqrt{-1}\theta_2}, 0) \left| \begin{array}{l} \cdot r_1 = a_j^{-1} t^{1-m_j} r_2^{m_j}, \\ 0 \leq r_1 \leq b_1^t = t^{c_1}; \\ \cdot \theta_1 = m_j \theta_2 \pmod{2\pi} \end{array} \right. \right\}.$$

The approximate metric is given by

$$ds^2 = \left(1 + m_j^2 a_j^{-2} t^{2(1-m_j)} r_2^{2(m_j-1)}\right) dr_2^2 + r_2^2 \left(1 + m_j^2 a_j^{-2} t^{2(1-m_j)} r_2^{2(m_j-1)}\right) d\theta_2^2 + du_3^2.$$

which gives the approximate volume-form:

$$vol^t = r_2 \left(1 + m_j^2 a_j^{-2} t^{2(1-m_j)} r_2^{2(m_j-1)}\right) dr_2 \wedge d\theta_2 \wedge du_3.$$

The imaginary part ε^t of the ratio calibration/volume-form is approximated by

$$\varepsilon^t = O(r_1) + O(r_2) = O(t^{1-m_j} r_2^{m_j}) + O(r_2).$$

and $|d\varepsilon^t|$ is approximated by

$$|d\varepsilon^t| = O(t^{1-m_j} r_2^{m_j-1}) + O(1).$$

(a.2) *Estimating $\|\varepsilon^t\|_{C^0}$ on P_j^t .* It follows from Part (a.1) (cf. Proposition 3.2.4) that

$$\|\varepsilon^t\|_{C^0} = O(t^{c_1}) + O(t^{(1-\frac{1}{m_j})+\frac{c_1}{m_j}}).$$

(a.3) *Estimating $\|\varepsilon^t\|_{L^{6/5}}$ on P_j^t .*

$$\begin{aligned} \|\varepsilon^t\|_{L^{6/5}} &= \left(2\pi l_j \int_{r_2=0}^{a^{1/m_j} t^{(m_j-1)/m_j} (b_1^t)^{1/m_j}} \left| O(t^{1-m_j} r_2^{m_j}) + O(r_2) \right|^{6/5} \right. \\ &\quad \cdot r_2 \left(1 + m_j^2 a_j^{-2} t^{2(1-m_j)} r_2^{2(m_j-1)} \right) dr_2 \left. \right)^{5/6} \\ &= O(1) \left(\int_{r_2=0}^{O(t^{1+\frac{c_1-1}{m_j}})} \left| O(t^{1-m_j} r_2^{m_j}) + O(r_2) \right|^{6/5} \right. \\ &\quad \cdot r_2 \left(1 + O(t^{2(1-m_j)} r_2^{2(m_j-1)}) \right) dr_2 \left. \right)^{5/6}. \end{aligned}$$

Note that both $O(t^{1-m_j} r_2^{m_j}) = O(r_2)$ and $1 = O(t^{2(1-m_j)} r_2^{2(m_j-1)})$ occur at $r_2 = O(t)$. Thus:
 If $0 < c_1 \leq 1$, then for $\delta > 0$ small enough, and all $t \in (0, \delta)$,

$$0 < O(t) \leq O(t^{1+\frac{c_1-1}{m_j}})$$

$$\begin{aligned} \|\varepsilon^t\|_{L^{6/5}} &= O(1) \left(\left(\int_{r_2=0}^{O(t)} + \int_{r_2=O(t)}^{O(t^{1+\frac{c_1-1}{m_j}})} \right) |O(t^{1-m_j} r_2^{m_j}) + O(r_2)|^{6/5} \right. \\ &\quad \left. \cdot r_2 \left(1 + O(t^{2(1-m_j)} r_2^{2(m_j-1)}) \right) dr_2 \right)^{5/6} \\ &= O(1) \left(\int_{r_2=0}^{O(t)} O(r_2)^{6/5} \cdot r_2 \cdot O(1) dr_2 \right. \\ &\quad \left. + \int_{r_2=O(t)}^{O(t^{1+\frac{c_1-1}{m_j}})} O(t^{1-m_j} r_2^{m_j})^{6/5} \cdot r_2 \cdot O(t^{2(1-m_j)} r_2^{2(m_j-1)}) dr_2 \right)^{5/6} \\ &= \left(O(t^{16/5}) + O(t^{16c_1/5}) \right)^{5/6} = O(t^{8c_1/3}). \end{aligned}$$

If $c_1 > 1$, then $O(t^{1+\frac{c_1-1}{m_j}}) < O(t)$ and

$$\|\varepsilon^t\|_{L^{6/5}} = \left(\int_{r_2=0}^{O(t^{1+\frac{c_1-1}{m_j}})} O(r_2)^{6/5} \cdot r_2 dr_2 \right)^{5/6} = O(t^{\frac{8}{3}(1+\frac{c_1-1}{m_j})}).$$

(a.4) *Estimating $\|\varepsilon^t\|_{L^1}$ on P_j^t .* Similar to the estimation for $\|\varepsilon^t\|_{L^{6/5}}$ in Part (a.3), if $0 < c_1 \leq 1$, then

$$\begin{aligned} \|\varepsilon^t\|_{L^1} &= \int_{r_2=0}^{O(t)} O(r_2) \cdot r_2 \cdot O(1) dr_2 \\ &\quad + \int_{r_2=O(t)}^{O(t^{1+\frac{c_1-1}{m_j}})} O(t^{1-m_j} r_2^{m_j}) \cdot r_2 \cdot O(t^{2(1-m_j)} r_2^{2(m_j-1)}) dr_2 \\ &= O(t^3) + O(t^{3c_1}) = O(t^{3c_1}). \end{aligned}$$

If $c_1 > 1$, then

$$\|\varepsilon^t\|_{L^1} = \int_{r_2=0}^{O(t^{1+\frac{c_1-1}{m_j}})} O(r_2) \cdot r_2 \cdot O(1) dr_2 = O(t^{3(1+\frac{c_1-1}{m_j})}).$$

(a.5) *Estimating $\|d\varepsilon^t\|_{L^6}$ on P_j^t .* Note $O(t^{1-m_j} r_2^{m_j-1}) = O(1)$ occurs also at $r_2 = O(t)$. Thus,

similar to the estimation for $\|\varepsilon^t\|_{L^{6/5}}$ in Part (a.3), if $0 < c_1 \leq 1$, then

$$\begin{aligned} \|d\varepsilon^t\|_{L^6} &= \left(\int_{r_2=0}^{O(t)} O(1)^6 \cdot r_2 \cdot O(1) dr_2 \right. \\ &\quad \left. + \int_{r_2=O(t)}^{O(t^{1+\frac{c_1-1}{m_j}})} O(t^{1-m_j} r_2^{m_j-1})^6 \cdot r_2 \cdot O(t^{2(1-m_j)} r_2^{2(m_j-1)}) dr_2 \right)^{1/6} \\ &= \left(O(t^2) + O(t^{2c_1(4-\frac{3}{m_j})+6(\frac{1}{m_j}-1)}) \right)^{1/6} = O(t^{c_1(\frac{4}{3}-\frac{1}{m_j})+(\frac{1}{m_j}-1)}). \end{aligned}$$

If $c_1 > 1$, then

$$\|d\varepsilon^t\|_{L^1} = \left(\int_{r_2=0}^{O(t^{1+\frac{c_1-1}{m_j}})} O(1)^6 \cdot r_2 \cdot O(1) dr_2 \right)^{1/6} = O(t^{\frac{1}{3}(1+\frac{c_1-1}{m_j})}).$$

(b) On Q_j^t .

(b.1) Basic data for estimates on Q_j^t . Consider

$$f^t(Q_j^t) = \left\{ (r_1 e^{\sqrt{-1}\theta_1}, u_3, r_2 e^{\sqrt{-1}\theta_2}, 0) \left| \begin{array}{l} \cdot r_2 = a_j^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right], \\ b_1^t = t^{c_1} \leq r_1 \leq t^{c_2} = b_2^t; \\ \cdot m_j \theta_2 = \theta_1 \pmod{2\pi} \end{array} \right. \right\}.$$

The approximate metric is given by

$$\begin{aligned} ds^2 &= \left(1 + a_j^{2/m_j} t^{2(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \dot{\chi}^t(r_1) r_1^{1/m_j} + \frac{1}{m_j} \chi^t(r_1) r_1^{(1-m_j)/m_j} \right]^2 \right) dr_1^2 \\ &\quad + \left(r_1^2 + m_j^{-2} a_j^{2/m_j} t^{2(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right]^2 \right) d\theta_1^2 + du_3^2, \end{aligned}$$

which gives the approximate volume-form

$$\begin{aligned} vol^t &= \left(1 + a_j^{2/m_j} t^{2(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \dot{\chi}^t(r_1) r_1^{1/m_j} + \frac{1}{m_j} \chi^t(r_1) r_1^{(1-m_j)/m_j} \right]^2 \right)^{1/2} \\ &\quad \cdot \left(r_1^2 + m_j^{-2} a_j^{2/m_j} t^{2(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right]^2 \right)^{1/2} dr_1 \wedge d\theta_1 \wedge du_3. \end{aligned}$$

Thus,

$$\begin{aligned} |vol^t| &\leq \left(1 + O(t^{2(1-\frac{1}{m_j})}) \left[O(t^{-2c_2}) + O(t^{-c_2} r_1^{\frac{1}{m_j}}) + O(r_1^{\frac{1}{m_j}-1}) \right]^2 \right)^{1/2} \\ &\quad \cdot \left(r_1^2 + O(t^{2(1-\frac{1}{m_j})}) \left[O(t^{-c_2}) + O(r_1^{\frac{1}{m_j}}) \right]^2 \right)^{1/2} |dr_1 \wedge d\theta_1 \wedge du_3| \\ &= \left(1 + O(t^{2(1-\frac{1}{m_j})}) \left[O(t^{-2c_2}) + O(r_1^{\frac{1}{m_j}-1}) \right]^2 \right)^{1/2} \\ &\quad \cdot \left(r_1^2 + O(t^{2(1-\frac{1}{m_j}-c_2)}) \right)^{1/2} |dr_1 \wedge d\theta_1 \wedge du_3| \end{aligned}$$

The imaginary part ε^t of the ratio calibration/volume-form is approximated by

$$\begin{aligned}\varepsilon^t &= O(r_2) + O(dr_2/dr_1) \\ &= O\left(t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r_1) + \chi^t(r_1) r_1^{1/m_j} \right] \right) \\ &\quad + O\left(t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \dot{\chi}^t(r_1) r_1^{1/m_j} + \frac{1}{m_j} \chi^t(r_1) r_1^{(1-m_j)/m_j} \right] \right); \end{aligned}$$

Thus,

$$\begin{aligned}|\varepsilon^t| &\leq O(t^{1-\frac{1}{m_j}}) \left[O(t^{-2c_2}) + O(t^{-c_2} r_1^{\frac{1}{m_j}}) + O(r_1^{\frac{1}{m_j}-1}) \right] \\ &= O(t^{1-\frac{1}{m_j}}) \left[O(t^{-2c_2}) + O(r_1^{\frac{1}{m_j}-1}) \right]. \end{aligned}$$

$|d\varepsilon^t|$ is approximated by

$$\begin{aligned}|d\varepsilon^t| &= O(dr_2/dr_1) + O(d^2r_2/dr_1^2) \\ &= O\left(t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \dot{\chi}^t(r_1) r_1^{1/m_j} + \frac{1}{m_j} \chi^t(r_1) r_1^{(1-m_j)/m_j} \right] \right) \\ &\quad + O\left(t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \ddot{\chi}^t(r_1) + \ddot{\chi}^t(r_1) r_1^{1/m_j} \right. \right. \\ &\quad \left. \left. + \frac{2}{m_j} \dot{\chi}^t(r_1) r_1^{(1-m_j)/m_j} + \frac{1-m_j}{m_j^2} \chi^t(r_1) r_1^{(1-2m_j)/m_j} \right] \right) \\ &\leq O(t^{1-\frac{1}{m_j}}) \left[O(t^{-3c_2}) + O(t^{-2c_2} r_1^{\frac{1}{m_j}}) + O(t^{-c_2} r_1^{\frac{1}{m_j}-1}) + O(r_1^{\frac{1}{m_j}-2}) \right] \\ &= O(t^{1-\frac{1}{m_j}}) \left[O(t^{-3c_2}) + O(t^{-c_2} r_1^{\frac{1}{m_j}-1}) + O(r_1^{\frac{1}{m_j}-2}) \right] \\ &= O(t^{1-\frac{1}{m_j}}) \left[O(t^{-3c_2}) + O(r_1^{\frac{1}{m_j}-2}) \right]. \end{aligned}$$

Here, we use $O(t^{-c_2} r_1^{\frac{1}{m_j}-1}) + O(r_1^{\frac{1}{m_j}-2}) = O(r_1^{\frac{1}{m_j}-2})$ for $0 < r_1 < t^{c_2}$ in the last equality.

(b.2) *Estimating $\|\varepsilon^t\|_{C^0}$ on Q_j^t .* It follows from Part (b.1) (cf. Proposition 3.2.4) that

$$\|\varepsilon^t\|_{C^0} \leq O(t^{1-\frac{1}{m_j}-2c_2}) + O(t^{(1-\frac{1}{m_j})(1-c_1)}).$$

(b.3) *Estimating $\|\varepsilon^t\|_{L^{6/5}}$ on Q_j^t .* It follows from Part (b.1) that

$$\begin{aligned}\|\varepsilon^t\|_{L^{6/5}} &\leq \left(2\pi m_j l_j \int_{r_1=t^{c_1}}^{t^{c_2}} \left| O(t^{1-\frac{1}{m_j}}) \left[O(t^{-2c_2}) + O(r_1^{\frac{1}{m_j}-1}) \right] \right|^{6/5} \right. \\ &\quad \cdot \left(1 + O(t^{2(1-\frac{1}{m_j})}) \left[O(t^{-2c_2}) + O(r_1^{\frac{1}{m_j}-1}) \right]^2 \right)^{1/2} \\ &\quad \left. \cdot \left(r_1^2 + O(t^{2(1-\frac{1}{m_j}-c_2)}) \right)^{1/2} dr_1 \right)^{5/6}. \end{aligned}$$

The equalities

$$\begin{aligned} O(t^{-2c_2}) &= O(r_1^{\frac{1}{m_j}-1}); & 1 &= O(t^{2(1-\frac{1}{m_j}-2c_2)}), \\ 1 &= O(t^{2(1-\frac{1}{m_j})} r_1^{2(\frac{1}{m_j}-1)}); & r_1^2 &= O(t^{2(1-\frac{1}{m_j}-c_2)}) \end{aligned}$$

are solved by

$$r_1 = O(t^{\frac{2c_2 m_j}{m_j-1}}); \quad c_2 = \frac{1}{2}(1 - \frac{1}{m_j}), \quad r_1 = O(t); \quad r_1 = O(t^{1-\frac{1}{m_j}-c_2})$$

respectively. Denote the t -exponents:

$$E_1 := c_1, \quad E_2 := c_2, \quad E_3 := \frac{2c_2 m_j}{m_j-1}, \quad E_4 := 1 - \frac{1}{m_j} - c_2, \quad E_5 := 1.$$

Then the equations $E_i = E_j$, $1 \leq i < j \leq 5$, divides $\{(c_1, c_2) \in (\mathbb{R}_+)^2 : c_1 > c_2\}$ into 13 regions:

region	description in $\{0 < c_2 < c_1\}$	order
<i>Region</i> ₍₁₎	$1 \leq c_2 (< c_1), \quad c_2 \geq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_5 \leq E_2 \leq E_1 \leq E_3$
<i>Region</i> ₍₂₎	$1 \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_5 \leq E_2 \leq E_3 \leq E_1$
<i>Region</i> ₍₃₎	$\frac{1}{2}(1 - \frac{1}{m_j}) \leq c_2 \leq 1, \quad c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_2 \leq E_5 \leq E_3 \leq E_1$
<i>Region</i> ₍₄₎	$c_1 \geq 1, \quad \frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})$	$E_2 \leq E_4 \leq E_3 \leq E_5 \leq E_1$
<i>Region</i> ₍₅₎	$c_1 \geq 1, \quad 0 < c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}$	$E_2 \leq E_3 \leq E_4 \leq E_5 \leq E_1$
<i>Region</i> ₍₆₎	$c_1 \geq 1, \quad \frac{1}{2}(1 - \frac{1}{m_j})c_1 \leq c_2 \leq 1$	$E_4 \leq E_2 \leq E_5 \leq E_1 \leq E_3$
<i>Region</i> ₍₇₎	$c_1 \leq 1, \quad \frac{1}{2}(1 - \frac{1}{m_j}) \leq c_2 (< c_1)$	$E_4 \leq E_2 \leq E_1 \leq E_5 \leq E_3$
<i>Region</i> ₍₈₎	$(1 - \frac{1}{m_j}) - c_1 \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j}), \quad c_2 \geq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_2 \leq E_4 \leq E_1 \leq E_3 \leq E_5$
<i>Region</i> ₍₉₎	$c_1 \leq 1, \quad \frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_2 \leq E_4 \leq E_3 \leq E_1 \leq E_5$
<i>Region</i> ₍₁₀₎	$c_1 \leq 1, \quad (0 <) c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}, \quad c_2 \geq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E_3 \leq E_4 \leq E_1 \leq E_5$
<i>Region</i> ₍₁₁₎	$\frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 (< c_1), \quad c_2 \leq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E_1 \leq E_4 \leq E_3 \leq E_5$
<i>Region</i> ₍₁₂₎	$\frac{1}{2}(1 - \frac{1}{m_j})c_1 \leq c_2 (< c_1), \quad c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}$	$E_2 \leq E_1 \leq E_3 \leq E_4 \leq E_5$
<i>Region</i> ₍₁₃₎	$(0 <) c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1, \quad c_2 \leq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E_3 \leq E_1 \leq E_4 \leq E_5$

Once the region (c_1, c_2) lies is fixed, the decomposition of the integral $\int_{r_1=t}^{c_2}$ is determined by the order of E_1, E_2, E_3, E_4, E_5 ; and for each part of the integral, the dominant t -order term in each factor of the integrand is determined. Similar computations as those in Part (a) give then the following bounds for $\|\epsilon^t\|_{L^{6/5}}$ on Q_j^t :

(c_1, c_2)	$\ \varepsilon^t\ _{L^{6/5}} \leq \bullet$
$Region_{(1)}$	$O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2})$
$Region_{(2)}$	$O(t^{(\frac{8}{3}-\frac{11}{6}c_1)(1-\frac{1}{m_j})+\frac{5}{6}(c_1-c_2)}) + O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2})$
$Region_{(3)}$	$O(t^{(\frac{8}{3}-\frac{11}{6}c_1)(1-\frac{1}{m_j})+\frac{5}{6}(c_1-c_2)}) + O(t^{\frac{7}{2}-\frac{8}{3m_j}-\frac{9}{2}c_2}) + O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2})$
$Region_{(4)}$	$O(t^{(\frac{8}{3}-\frac{11}{6}c_1)(1-\frac{1}{m_j})+\frac{5}{6}(c_1-c_2)}) + O(t^{\frac{11}{6}(1-\frac{1}{m_j})+(\frac{5}{6}-2m_j+\frac{5}{3(m_j-1)})c_2})$ $+ O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{c_2}{3}}) \quad \text{for } 2 \leq m_j \leq 5,$ $O(t^{(\frac{8}{3}-\frac{11}{6}c_1)(1-\frac{1}{m_j})+\frac{5}{6}(c_1-c_2)}) + O(t^{\frac{11}{6}(1-\frac{1}{m_j}-\frac{5}{6}c_2)} \log t ^{5/6})$ $+ O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{c_2}{3}}) \quad \text{for } m_j = 6,$ $O(t^{(\frac{8}{3}-\frac{11}{6}c_1)(1-\frac{1}{m_j})+\frac{5}{6}(c_1-c_2)}) + O(t^{\frac{5}{6}(2-\frac{1}{m_j}-c_2)})$ $+ O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{c_2}{3}}) \quad \text{for } m_j \geq 7$
$Region_{(5)}$	$O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{5}{6}c_2+(\frac{11}{6m_j}-1)c_1}) + O(t^{\frac{5}{3}-\frac{5}{6m_j}-\frac{5}{6}c_2})$ $+ O(t^{\frac{11}{6}(1-\frac{1}{m_j})+(\frac{11-m_j)c_2}{3(m_j-1)}}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2}) \quad \text{for } 2 \leq m_j \leq 10,$ $O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{5}{6}c_2+(\frac{11}{6m_j}-1)c_1}) + O(t^{\frac{5}{3}-\frac{5}{6m_j}-\frac{5}{6}c_2})$ $+ O(t^{5/3} \log t ^{5/6}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2}) \quad \text{for } m_j = 11,$ $O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{5}{6}c_2+(\frac{11}{6m_j}-1)c_1}) + O(t^{\frac{5}{3}-\frac{5}{6m_j}-\frac{5}{6}c_2})$ $+ O(t^{\frac{1}{6}(1-\frac{1}{m_j})(10+\frac{11}{m_j})+\frac{1}{6}(1-\frac{11}{m_j})c_2}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2}) \quad \text{for } m_j \geq 12$
$Region_{(6)}$	$O(t^{\frac{7}{2}-\frac{8}{3m_j}-\frac{9}{2}c_2}) + O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2})$
$Region_{(7)}$	$O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2})$
$Region_{(8)}$	$O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$
$Region_{(9)}$	$O(t^{\frac{11}{6}(1-\frac{1}{m_j})-\frac{17}{6}c_2+\frac{5m_jc_2}{3(m_j-1)}}) + O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$ $\quad \text{for } 2 \leq m_j \leq 5,$ $O(t^{\frac{11}{6}(1-\frac{1}{m_j})-\frac{5}{6}c_2} \log t ^{5/6}) + O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$ $\quad \text{for } m_j = 6,$ $O(t^{\frac{11}{6}(1-\frac{1}{m_j})-(\frac{1}{m_j}-\frac{1}{6})c_1-\frac{5}{6}c_2}) + O(t^{\frac{8}{3}(1-\frac{1}{m_j})-\frac{11}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$ $\quad \text{for } m_j \geq 7$

$Region_{(10)}$	$O(t^{\frac{8}{3}(1-\frac{1}{m_j})-(1-\frac{1}{m_j})^2-(\frac{2}{3}+\frac{1}{m_j})c_2}) + O(t^{1-\frac{1}{m_j}+(\frac{1}{m_j}+\frac{2}{3})\frac{2c_2m_j}{m_j-1}}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$ <p style="text-align: center;"><i>for</i> $2 \leq m_j \leq 5$,</p> $O(t^{\frac{55}{36}-\frac{5}{6}c_2} \log t ^{5/6}) + O(t^{1-\frac{1}{m_j}+(\frac{1}{m_j}+\frac{2}{3})\frac{2c_2m_j}{m_j-1}}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$ <p style="text-align: center;"><i>for</i> $m_j = 6$,</p> $O(t^{\frac{11}{6}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{1}{6})c_1-\frac{5}{6}c_2}) + O(t^{1-\frac{1}{m_j}+(\frac{1}{m_j}+\frac{2}{3})\frac{2c_2m_j}{m_j-1}}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$ <p style="text-align: center;"><i>for</i> $m_j \geq 7$</p>
$Region_{(11)}$	$O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$
$Region_{(12)}$	$O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$
$Region_{(13)}$	$O(t^{1-\frac{1}{m_j}+\frac{4}{3}c_2+\frac{10c_2}{3(m_j-1)}}) + O(t^{1-\frac{1}{m_j}-\frac{1}{3}c_2})$

(b.4) *Estimating $\|\varepsilon^t\|_{L^1}$ on Q_j^t .* Except an adjustment of powers due the change from $L^{6/5}$ -norm to L^1 -norm, the computations in this case is completely the same as those in Part (b.3):

(c_1, c_2)	$\ \varepsilon^t\ _{L^1} \leq \bullet$
$Region_{(1)}$	$O(t^{3(1-\frac{1}{m_j})-4c_2})$
$Region_{(2)}$	$O(t^{\frac{3}{2}-c_2} \log t) + O(t^{3(1-\frac{1}{m_j})-4c_2})$ for $m_j = 2$, $O(t^{3(1-\frac{1}{m_j})+(\frac{2}{m_j}-1)c_1-c_2}) + O(t^{3(1-\frac{1}{m_j})-4c_2})$ for $m_j \geq 3$
$Region_{(3)}$	$O(t^{\frac{3}{2}-c_2} \log t) + O(t^{4-\frac{3}{m_j}-5c_2}) + O(t^{3(1-\frac{1}{m_j})-4c_2})$ for $m_j = 2$, $O(t^{3(1-\frac{1}{m_j})+(\frac{2}{m_j}-1)c_1-c_2}) + O(t^{4-\frac{3}{m_j}-5c_2}) + O(t^{3(1-\frac{1}{m_j})-4c_2})$ for $m_j \geq 3$
$Region_{(4)}$	$O(t^{\frac{3}{2}-c_2} \log t) + O(t^{2(1-\frac{1}{m_j})-c_2+\frac{2c_2}{m_j-1}}) + O(t^{3(1-\frac{1}{m_j})-4c_2}) + O(t^{1-\frac{1}{m_j}})$ for $m_j = 2$, $O(t^{3(1-\frac{1}{m_j})+(\frac{2}{m_j}-1)c_1-c_2}) + O(t^{2(1-\frac{1}{m_j})-c_2+\frac{2c_2}{m_j-1}}) + O(t^{3(1-\frac{1}{m_j})-4c_2}) + O(t^{1-\frac{1}{m_j}})$ for $m_j \geq 3$
$Region_{(5)}$	$O(t^{\frac{3}{2}-c_2} \log t) + O(t^{2(1-\frac{1}{m_j})+\frac{4c_2}{m_j-1}}) + O(t^{1-\frac{1}{m_j}})$ for $m_j = 2$, $O(t^{3(1-\frac{1}{m_j})+(\frac{2}{m_j}-1)c_1-c_2}) + O(t^{2-\frac{1}{m_j}-c_2}) + O(t^{2(1-\frac{1}{m_j})+\frac{4c_2}{m_j-1}}) + O(t^{1-\frac{1}{m_j}})$ for $m_j \geq 3$
$Region_{(6)}$	$O(t^{4-\frac{3}{m_j}-5c_2}) + O(t^{3(1-\frac{1}{m_j})-4c_2})$
$Region_{(7)}$	$O(t^{3(1-\frac{1}{m_j})-4c_2})$
$Region_{(8)}$	$O(t^{3(1-\frac{1}{m_j})-4c_2}) + O(t^{1-\frac{1}{m_j}})$
$Region_{(9)}$	$O(t^{2(1-\frac{1}{m_j})-c_2-\frac{2c_2}{m_j-1}}) + O(t^{3(1-\frac{1}{m_j})-4c_2}) + O(t^{1-\frac{1}{m_j}})$
$Region_{(10)}$	$O(t^{2-(1+\frac{1}{m_j})(c_2+\frac{1}{m_j})}) + O(t^{1-\frac{1}{m_j}+\frac{2(m_j+1)}{m_j-1}c_2}) + O(t^{1-\frac{1}{m_j}})$
$Region_{(11)}$	$O(t^{1-\frac{1}{m_j}})$
$Region_{(12)}$	$O(t^{1-\frac{1}{m_j}})$
$Region_{(13)}$	$O(t^{1-\frac{1}{m_j}+\frac{2(m_j+1)}{m_j-1}c_2}) + O(t^{1-\frac{1}{m_j}})$

(b.5) *Estimating $\|d\varepsilon^t\|_{L^6}$ on Q_j^t .* It follows from Part (b.1) that

$$\begin{aligned} \|\varepsilon^t\|_{L^6} \leq & \left(2\pi m_j l_j \int_{r_1=t^{c_1}}^{t^{c_2}} \left| O(t^{1-\frac{1}{m_j}}) \left[O(t^{-3c_2}) + O(r_1^{\frac{1}{m_j}-2}) \right] \right|^6 \right. \\ & \cdot \left(1 + O(t^{2(1-\frac{1}{m_j})}) \left[O(t^{-2c_2}) + O(r_1^{\frac{1}{m_j}-1}) \right]^2 \right)^{1/2} \\ & \left. \cdot \left(r_1^2 + O(t^{2(1-\frac{1}{m_j}-c_2)}) \right)^{1/2} dr_1 \right)^{1/6}. \end{aligned}$$

Similar to the discussion in Part (b.3), the equalities

$$\begin{aligned} O(t^{-3c_2}) &= O(r_1^{\frac{1}{m_j}-2}); & O(t^{-2c_2}) &= O(r_1^{\frac{1}{m_j}-1}), \\ 1 &= O(t^{2(1-\frac{1}{m_j}-2c_2)}), & 1 &= O(t^{2(1-\frac{1}{m_j})} r_1^{2(\frac{1}{m_j}-1)}); & r_1^2 &= O(t^{2(1-\frac{1}{m_j}-c_2)}) \end{aligned}$$

are solved by

$$r_1 = O(t^{\frac{3c_2 m_j}{2m_j-1}}); \quad r_1 = O(t^{\frac{2c_2 m_j}{m_j-1}}); \quad c_2 = \frac{1}{2}(1 - \frac{1}{m_j}), \quad r_1 = O(t); \quad r_1 = O(t^{1-\frac{1}{m_j}-c_2})$$

respectively. Recall/denote the t -exponents:

$$E_1 := c_1, \quad E_2 := c_2, \quad E_3 := \frac{2c_2 m_j}{m_j-1}, \quad E'_3 := \frac{3c_2 m_j}{2m_j-1}, \quad E_4 := 1 - \frac{1}{m_j} - c_2, \quad E_5 := 1.$$

Then the equations E_i (or E'_3) = E_j (or E'_3), $1 \leq i < j \leq 5$, refine the previous 13-region decomposition of $\{(c_1, c_2) \in (\mathbb{R}_+)^2 : c_1 > c_2\}$ further into 26 regions:

<i>region</i>	<i>description in $\{0 < c_2 < c_1\}$</i>	<i>order</i>
$Region_{(1)1}$	$1 \leq c_2 (< c_1), \quad c_2 \geq \frac{2}{3}(1 - \frac{1}{2m_j})c_1$	$E_4 \leq E_5 \leq E_2 \leq E_1 \leq E'_3 \leq E_3$
$Region_{(1)2}$	$1 \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j})c_1, \quad c_2 \geq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_5 \leq E_2 \leq E'_3 \leq E_1 \leq E_3$
$Region_{(2)}$	$1 \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_5 \leq E_2 \leq E'_3 \leq E_3 \leq E_1$
$Region_{(3)1}$	$\frac{2}{3}(1 - \frac{1}{2m_j}) \leq c_2 \leq 1, \quad c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_2 \leq E_5 \leq E'_3 \leq E_3 \leq E_1$
$Region_{(3)2}$	$\frac{1}{2}(1 - \frac{1}{m_j}) \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j}), \quad c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_4 \leq E_2 \leq E'_3 \leq E_5 \leq E_3 \leq E_1$
$Region_{(4)1}$	$c_1 \geq 1, \quad \frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)} \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})$	$E_2 \leq E_4 \leq E'_3 \leq E_3 \leq E_5 \leq E_1$
$Region_{(4)2}$	$c_1 \geq 1, \quad \frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 \leq \frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)}$	$E_2 \leq E'_3 \leq E_4 \leq E_3 \leq E_5 \leq E_1$
$Region_{(5)}$	$c_1 \geq 1, \quad 0 < c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}$	$E_2 \leq E'_3 \leq E_3 \leq E_4 \leq E_5 \leq E_1$
$Region_{(6)1}$	$c_1 \geq 1, \quad \frac{2}{3}(1 - \frac{1}{2m_j})c_1 \leq c_2 \leq 1$	$E_4 \leq E_2 \leq E_5 \leq E_1 \leq E'_3 \leq E_3$
$Region_{(6)2}$	$\frac{2}{3}(1 - \frac{1}{2m_j}) \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j})c_1, \quad \frac{1}{2}(1 - \frac{1}{m_j})c_1 \leq c_2 \leq 1$	$E_4 \leq E_2 \leq E_5 \leq E'_3 \leq E_1 \leq E_3$
$Region_{(6)3}$	$c_1 \geq 1, \quad \frac{1}{2}(1 - \frac{1}{m_j})c_1 \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j})$	$E_4 \leq E_2 \leq E'_3 \leq E_5 \leq E_1 \leq E_3$
$Region_{(7)1}$	$c_1 \leq 1, \quad \frac{2}{3}(1 - \frac{1}{2m_j}) \leq c_2 (< c_1)$	$E_4 \leq E_2 \leq E_1 \leq E_5 \leq E'_3 \leq E_3$
$Region_{(7)2}$	$\frac{2}{3}(1 - \frac{1}{2m_j})c_1 \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j}), \quad \frac{1}{2}(1 - \frac{1}{m_j}) \leq c_2 (< c_1)$	$E_4 \leq E_2 \leq E_1 \leq E'_3 \leq E_5 \leq E_3$
$Region_{(7)3}$	$c_1 \leq 1, \quad \frac{1}{2}(1 - \frac{1}{m_j}) \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j})c_1$	$E_4 \leq E_2 \leq E'_3 \leq E_1 \leq E_5 \leq E_3$
$Region_{(8)1}$	$(1 - \frac{1}{m_j}) - c_1 \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j}), \quad c_2 \geq \frac{2}{3}(1 - \frac{1}{2m_j})c_1$	$E_2 \leq E_4 \leq E_1 \leq E'_3 \leq E_3 \leq E_5$
$Region_{(8)2}$	$\frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)} \leq c_2 \leq \frac{2}{3}(1 - \frac{1}{2m_j})c_1, \quad \frac{1}{2}(1 - \frac{1}{m_j})c_1 \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})$	$E_2 \leq E_4 \leq E'_3 \leq E_1 \leq E_3 \leq E_5$
$Region_{(8)3}$	$(1 - \frac{1}{m_j}) - c_1 \leq c_2 \leq \frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)}, \quad c_2 \geq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_2 \leq E'_3 \leq E_4 \leq E_1 \leq E_3 \leq E_5$
$Region_{(9)1}$	$c_1 \leq 1, \quad \frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)} \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1$	$E_2 \leq E_4 \leq E'_3 \leq E_3 \leq E_1 \leq E_5$
$Region_{(9)2}$	$c_1 \leq 1, \quad \frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1, \quad c_2 \leq \frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)}$	$E_2 \leq E'_3 \leq E_4 \leq E_3 \leq E_1 \leq E_5$
$Region_{(10)}$	$c_1 \leq 1, \quad (0 <) c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}, \quad c_2 \geq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E'_3 \leq E_3 \leq E_4 \leq E_1 \leq E_5$
$Region_{(11)1}$	$\frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)} \leq c_2 (< c_1), \quad c_2 \leq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E_1 \leq E_4 \leq E'_3 \leq E_3 \leq E_5$
$Region_{(11)2}$	$\frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 (< c_1), \quad \frac{2}{3}(1 - \frac{1}{2m_j})c_1 \leq c_2 \leq \frac{(m_j-1)(2m_j-1)}{m_j(5m_j-1)}$	$E_2 \leq E_1 \leq E'_3 \leq E_4 \leq E_3 \leq E_5$
$Region_{(11)3}$	$\frac{(m_j-1)^2}{m_j(3m_j-1)} \leq c_2 < \frac{2}{3}(1 - \frac{1}{2m_j}), \quad c_2 \leq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E'_3 \leq E_1 \leq E_4 \leq E_3 \leq E_5$
$Region_{(12)1}$	$\frac{2}{3}(1 - \frac{1}{2m_j})c_1 \leq c_2 (< c_1), \quad c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}$	$E_2 \leq E_1 \leq E'_3 \leq E_3 \leq E_4 \leq E_5$
$Region_{(12)2}$	$\frac{1}{2}(1 - \frac{1}{m_j})c_1 \leq c_2 < \frac{2}{3}(1 - \frac{1}{2m_j})c_1, \quad c_2 \leq \frac{(m_j-1)^2}{m_j(3m_j-1)}$	$E_2 \leq E'_3 \leq E_1 \leq E_3 \leq E_4 \leq E_5$
$Region_{(13)}$	$(0 <) c_2 \leq \frac{1}{2}(1 - \frac{1}{m_j})c_1, \quad c_2 \leq (1 - \frac{1}{m_j}) - c_1$	$E_2 \leq E'_3 \leq E_3 \leq E_1 \leq E_4 \leq E_5$

Similar computations as those in Part (b.3) gives then the following bounds for $\|d\varepsilon^t\|_{L^6}$ on Q_j^t :

(c_1, c_2)	$\ d\varepsilon^t\ _{L^6} \leq \bullet$
$Region_{(1)_1}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(1)_2}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(2)}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{7}{6m_j}-2)c_1-\frac{1}{6}c_2})$ $+ O(t^{\frac{4}{3}(1-\frac{1}{m_j})-(\frac{25}{6}+\frac{5}{3(m_j-1)})c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(3)_1}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{7}{6m_j}-2)c_1-\frac{1}{6}c_2}) + O(t^{-\frac{25}{6}-\frac{5}{3(m_j-1)}})$ $+ O(t^{\frac{3}{2}-\frac{4}{3m_j}-\frac{7}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(3)_2}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{6}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-(\frac{25}{6}+\frac{5}{3(m_j-1)})c_2})$ $+ O(t^{-\frac{1}{2}-\frac{1}{3m_j}-\frac{1}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{23}{6}c_2})$
$Region_{(4)_1}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{7}{6m_j}-2)c_1-\frac{1}{6}c_2}) + O(t^{-\frac{2}{3}-\frac{1}{6m_j}-\frac{1}{6}c_2})$ $+ O(t^{\frac{7}{6}(1-\frac{1}{m_j})-(\frac{23}{6}+\frac{5}{3(m_j-1)})c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(4)_2}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{7}{6m_j}-2)c_1-\frac{1}{6}c_2}) + O(t^{-\frac{2}{3}-\frac{1}{6m_j}-\frac{1}{6}c_2})$ $+ O(t^{\frac{7}{6}(1-\frac{1}{m_j})-(\frac{23}{6}+\frac{5}{3(m_j-1)})c_2}) + O(t^{(1-\frac{1}{m_j})(\frac{1}{m_j}-\frac{2}{3})-(\frac{1}{m_j}-\frac{5}{3})c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(5)}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{7}{6m_j}-2)c_1-\frac{1}{6}c_2}) + O(t^{-\frac{2}{3}-\frac{1}{6m_j}-\frac{1}{6}c_2})$ $+ O(t^{(1-\frac{1}{m_j})(\frac{7}{6m_j}-\frac{2}{3})-(\frac{7}{6m_j}-\frac{11}{6})c_2}) + O(t^{(1-\frac{1}{m_j})-\frac{2(5m_j-3)}{3(m_j-1)}}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(6)_1}$	$O(t^{\frac{3}{2}-\frac{4}{3m_j}-\frac{7}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(6)_2}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{2}c_2}) + O(t^{\frac{3}{2}-\frac{4}{3m_j}-\frac{7}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(6)_3}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{2}c_2}) + O(t^{-\frac{1}{2}-\frac{1}{3m_j}-\frac{1}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(7)_1}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(7)_2}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(7)_3}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{2}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2})$
$Region_{(8)_1}$	$O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{7}{2}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(8)_2}$	$O(t^{\frac{7}{6}(1-\frac{1}{m_j})+\frac{9-35m_j}{6(2m_j-1)}c_2}) + O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(8)_3}$	$O(t^{\frac{7}{6}(1-\frac{1}{m_j})+\frac{9-35m_j}{6(2m_j-1)}c_2}) + O(t^{(1-\frac{1}{m_j})(\frac{1}{m_j}-\frac{2}{3})+(\frac{1}{m_j}-\frac{5}{3})c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$

$Region_{(9)_1}$	$O(t^{\frac{7}{6}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{6}c_2}) + O(t^{\frac{7}{6}(1-\frac{1}{m_j})-(\frac{23}{6}+\frac{5}{3(m_j-1)})c_2})$ $+ O(t^{\frac{4}{3}(1-\frac{1}{m_j})-\frac{10}{3}c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(9)_2}$	$O(t^{\frac{7}{6}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{6}c_2}) + O(t^{\frac{7}{6}(1-\frac{1}{m_j})-(\frac{23}{6}+\frac{5}{3(m_j-1)})c_2})$ $+ O(t^{(1-\frac{1}{m_j})(\frac{1}{m_j}-\frac{2}{3})+(\frac{1}{m_j}-\frac{5}{3})c_2}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(10)}$	$O(t^{\frac{7}{6}(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{11}{6})c_1-\frac{1}{6}c_2}) + O(t^{(1-\frac{1}{m_j})(\frac{1}{m_j}-\frac{2}{3})+(\frac{1}{m_j}-\frac{5}{3})c_2})$ $+ O(t^{(1-\frac{1}{m_j})-\frac{2(5m_j-3)}{3(m_j-1)c_2}}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(11)_1}$	$O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(11)_2}$	$O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(11)_3}$	$O(t^{(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{5}{3})c_1}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(12)_1}$	$O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(12)_2}$	$O(t^{(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{5}{3})c_1}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$
$Region_{(13)}$	$O(t^{(1-\frac{1}{m_j})+(\frac{1}{m_j}-\frac{5}{3})c_1}) + O(t^{(1-\frac{1}{m_j})-\frac{2(5m_j-3)}{3(m_j-1)c_2}}) + O(t^{1-\frac{1}{m_j}-\frac{8}{3}c_2})$

(c) *Overall estimates on X .* Adding the bound on P_j^t and the bound on Q_j^t , $j = 1, \dots, \tilde{n}_0$ gives a bound of Sobolev norms on X . The proposition follows from the explicit expressions in Part (a) and Part (b). \square

Remark 3.2.6. [further refinement]. Once having the explicit exponents in the t -powers, one can further refine each region in the (c_1, c_2) -plane in the proof so that in the end there is only one dominating t -power in each final region. As the details are straightforward but very tedious and there are issues one cannot bypass due to topological reasons (cf. Sec. 4), we omit the discussion of such further refinement.

3.3 Lagrangian neighborhoods and bounds on $R(g^t)$, $\delta(g^t)$.

Recall Definition 3.1.4, the Lagrangian neighborhood $\Phi_{Z_0} : U_{Z_0} \rightarrow Y$ of Z_0 in Y , and the symplectic covering map $f_1^\diamond : T^*X^\diamond \rightarrow T^*Z_0^\diamond$ in Sec. 3.1. In this subsection, we address how tractable it is to construct an immersed Lagrangian neighborhood for f^t by gluing an admissible Lagrangian neighborhood $\Phi_{\nu_X(\tilde{\Gamma}_j)}^{j,a_j,t} : U_{\nu_X(\tilde{\Gamma}_j)}^{a_j,t} \rightarrow Y^{t,j}$ of $L^{t,a_j}t^{m_j-1}$ under $\psi^{j,a_j}t^{m_j-1}$, constructed in Proposition 2.4.1, to $\Phi_{Z_0} \circ f_1^\diamond$ on a neighborhood of the zero-section of T^*X^\diamond so that the immersed Lagrangian neighborhood of f^t can fit into Joyce's criteria (iii), (iv), and (v).

Definition/Lemma 3.3.1. [admissible immersed Lagrangian neighborhood for $f^{t,\diamond}$]. Let $f^{t,\diamond} := f^t|_{X^\diamond}$. Define $f_1^{\diamond,+t} : T^*X^\diamond \rightarrow T^*Z_0^\diamond$ to be the symplectic covering map $f_1^{\diamond,+t}(\cdot) = f_1^\diamond(\cdot + dh^t)$ and recall that $\Phi_{Z_0} \circ f_1^{\diamond,+t}|_{\text{zero-section}} = f^{t,\diamond}$ from Definition 3.1.4. Assume that $b_2^t = Ct^{1-\eta}$ for some constants $C > 0$ and $0 < \eta < 1$. Then there exists an $\epsilon_0 > 0$ such that for $\delta > 0$ small enough and $t \in (0, \delta)$, the restriction $\Phi_{U_{X^\diamond}^t}$ of $f_1^{\diamond,+t}$ to the following neighborhood

$$U_{X^\diamond}^t := \{(x, v) \in T^*X^\diamond : |v|_{g^t} < t\epsilon_0\}$$

of the zero-section of T^*X^\diamond is a symplectic immersion into U_{Z_0} .

Proof. This follows from Proposition 2.4.4, the quasi-isometry nature of $\coprod_j \Upsilon^j : Y^{\iota,j} \rightarrow Y$, and

the following estimates of $1 / \sqrt{\left(1 + \left(\frac{d}{dr} \left(a^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r) + \chi^t(r) r^{1/m_j}\right]\right)\right)^2}$ over $tR'_0 \leq r \leq b_2^t$, cf. Remark 3.1.6.

First, note that

$$\begin{aligned} & \left| \frac{d}{dr} \left(a^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r) + \chi^t(r) r^{1/m_j} \right] \right) \right| \\ &= a^{1/m_j} t^{(m_j-1)/m_j} \left| \frac{m_j}{m_j+1} \ddot{\chi}^t(r) + \dot{\chi}^t(r) r^{1/m_j} + \frac{1}{m_j} \chi^t(r) r^{(1-m_j)/m_j} \right| \\ &\leq a^{1/m_j} t^{(m_j-1)/m_j} \left(\frac{m_j}{m_j+1} \cdot \frac{C_0}{(b_2^t)^2} + \frac{C_0}{b_2^t} r^{1/m_j} + \frac{1}{m_j} r^{(1-m_j)/m_j} \right) \\ &\leq O(t^{(m_j-1)/m_j}) (b_2^t)^{-2} + O(t^{(m_j-1)/m_j}) (b_2^t)^{(1-m_j)/m_j} + O(t^{(m_j-1)/m_j}) (tR'_0)^{(1-m_j)/m_j} \\ &= O(t^{(m_j-1)/m_j}) (b_2^t)^{-2} + O(t^{(m_j-1)/m_j}) (b_2^t)^{(1-m_j)/m_j} + O(1). \end{aligned}$$

Here, we have used the fact that r^{1/m_j} (resp. $r^{(1-m_j)/m_j}$) is an increasing (resp. decreasing) function over $r > 0$ for all j . Suppose that $b_2^t = O(t^{1-\eta})$ with $0 < \eta < 1$. Then,

$$\begin{aligned} & O(t^{(m_j-1)/m_j}) (b_2^t)^{-2} + O(t^{(m_j-1)/m_j}) (b_2^t)^{(1-m_j)/m_j} + O(1) \\ &= O(t^{-2(1-\eta)}) + O(t^{(1-\frac{1}{m_j})\eta}) + O(1) \leq O(t^{-2}). \end{aligned}$$

Consequently,

$$1 / \sqrt{\left(1 + \left(\frac{d}{dr} \left(a^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r) + \chi^t(r) r^{1/m_j}\right]\right)\right)^2} \geq O(t).$$

This implies that there exists an $\epsilon_0 > 0$ such that the following disk-bundle of constant radius

$$\{(x, v) \in T^*X^\diamond : \text{distance}_{g^t}(x, \tilde{\Gamma}) > tR'_0, |v|_{g^t} < \epsilon_0 t\}$$

is immersed into U_{Z_0} by $f_1^{\diamond,+,t}$. □

Definition 3.3.2. [immersed Lagrangian neighborhood for f^t via gluing]. Continuing the assumption of b_2^t in Definition/Lemma 3.3.1. Recall the constructions and notations in Sec. 2.4 and Sec. 3.1. Let $R'_0 > 0$ such that $0 < tR'_0 < b_1^t$. Performing the construction of Sec. 2.4 to $\psi^{j,a}$ for each j gives an admissible Lagrangian neighborhood $\Phi_{\nu_X(\tilde{\Gamma}_j)}^{j,a_j,t} : U_{\nu_X(\tilde{\Gamma}_j)}^{a_j,t} \rightarrow Y^{\iota,j}$ for $L'^{j,a_j} t^{m_j-1} \subset Y^{\iota,j}$ under $\psi^{j,a_j} t^{m_j-1}$. For $0 < r < R_0$, denote

$$\begin{aligned} \nu_{T^*X}(\tilde{\Gamma})_r &:= \coprod_j \{(x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) \in U_{\nu_X(\tilde{\Gamma}_j)}^{a_j,t} : |(x_1 + \sqrt{-1}x_2)^{m_j}| < r\} \\ &\subset \coprod_j T^*\nu_X(\tilde{\Gamma}_j). \end{aligned}$$

Then $\Phi_{\nu_X(\tilde{\Gamma}_j)}^{j,a_j,t}$ and $\Phi_{U_{X^\diamond}^t}$ coincide over

$$\begin{aligned} & \nu_{T^*X}(\tilde{\Gamma})_{b_1^t} \cap (U_{X^\diamond}^t - \overline{\nu_{T^*X}(\tilde{\Gamma})_{tR'_0}}) \\ &= \coprod_j \{(x_1, x_2, x_3, p_{x_1}, p_{x_2}, p_{x_3}) \in U_{\nu_X(\tilde{\Gamma}_j)}^{a_j,t} : tR'_0 < |(x_1 + \sqrt{-1}x_2)^{m_j}| < b_1^t\} \end{aligned}$$

under the built-in inclusions $U_{\nu_X(\tilde{\Gamma}_j)}^{a_j,t} \subset T^*X$ and $Y'^j \subset T^*Z_0$. It follows that if one takes the following neighborhood of the zero-section of T^*X :

$$U_X^t = \nu_{T^*X}(\tilde{\Gamma})_{b_1^t} \cup (U_{X^\diamond}^t - \overline{\nu_{T^*X}(\tilde{\Gamma})_{tR'_0}}) \quad \text{in } T^*X,$$

then the restrictions $\Phi_{\nu_X(\tilde{\Gamma}_j)}^{j,a_j,t} \Big|_{\nu_{T^*X}(\tilde{\Gamma})_{b_1^t}}$ and $\Phi_{U_{X^\diamond}^t} \Big|_{U_{X^\diamond}^t - \overline{\nu_{T^*X}(\tilde{\Gamma})_{tR'_0}}}$ glue to a symplectic immersion

$$\Phi_X^t : U_X^t \longrightarrow Y$$

whose restriction to the zero-section is f^t . Φ_X^t defines thus an immersed Lagrangian neighborhood for the immersed Lagrangian submanifold $f^t : X \rightarrow Y$.

Theorem 3.3.3. [bound for size, injective radius, and curvature]. *Recall Theorem 1.3 in Sec. 1. Making δ smaller if necessary, there exist $A_1, A_4, A_5, A_6 > 0$ such that the following hold for all $t \in (0, \delta)$:*

- (iii) *The subset $\mathcal{B}_{A_1 t} \subset T^*N^t$ of Definition 1.2 lies in U_{N^t} , and $\|\hat{\nabla}^k \beta^t\|_{C^0} \leq A_4 t^{-k}$ on $\mathcal{B}_{A_1 t}$ for $k = 0, 1, 2, 3$.*
- (iv) *The injectivity radius $\delta(g^t)$ satisfies $\delta(h^t) \geq A_5 t$.*
- (v) *The Riemann curvature $R(g^t)$ satisfies $\|R(g^t)\|_{C^0} \leq A_6 t^{-2}$.*

Proof. Except the necessary mild changes of details that are akin to particular situations, the proof of [Jo3: III. Theorem 6.8] applies here. In summary, note that the region in X where f^t and f differ has the image contained in $\coprod_j Y'^j$ for all t . Outside this region, f^t is independent of t and, hence, all the criteria in the theorem are satisfied. The theorem thus is a joint consequence of the following items:

- (1) Proposition 2.4.4, which leads to the size lower bound $A_1 t$ in Criterion (iii);
- (2) Proposition 2.4.6, which, together with Parts (3) and (5), leads to the bounds in Criterion (iii);
- (3) the estimate of $\left| \frac{d}{dr} \left(a^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r) + \chi^t(r) r^{1/m_j} \right] \right) \right|$ over $tR'_0 \leq r \leq b_2^t$ and the conclusion in the proof of Definition/Lemma 3.3.1, which is needed to control the behavior of $\hat{\nabla}$, and together with Parts (3) and (5), justifies Criteria (iv) and (v);
- (4) the quasi-isometry nature of $\coprod_j \Upsilon^j : Y'^j \rightarrow Y$, which is needed to relate the local study in the flat tangent space to that in Y .

□

3.4 Sobolev immersion inequalities on N^t .

In this subsection we discuss the (in)validity of the following statement in our situation:

Statement 3.4.1. [Sobolev immersion inequality]. (Cf. [Jo3: III. Theorem 6.12].) *Making $\delta > 0$ smaller if necessary, there exists $A_7 > 0$ such that for all $t \in (0, \delta)$, if $v \in L_1^2(N^t)$ with $\int_{N^t} v dV^t = 0$ then $v \in L^6(N^t)$ and $\|v\|_{L^6} \leq A_7 \|dv\|_{L^2}$.*

The investigation (and notations whenever possible) follows⁷ the setting of Joyce in [Jo3: III. Sec. 6.4]. See also [Bu1: Sec. 4.6, Sec. 5.3] of Adrian Butscher, [Lee: Sec. 3] of Yng-Ing Lee , and [Sa1: Sec. 4] of Sema Salur for related studies.

Preparation.

Some basic Sobolev inequalities are given here.

Theorem 3.4.2. [Michael-Simon inequality]. ([M-S] and [Jo3: III. Theorem 6.9].) *Assume that $m \geq 3$. Let $f : S \rightarrow \mathbb{R}^l$ be an immersed m -submanifold of \mathbb{R}^l and $u \in C_{cs}^1(S)$. Then $\|u\|_{L^{2m/(m-2)}} \leq D_1(\|u\|_{L^2} + \|uH\|_{L^2})$, where $D_1 > 0$ depends only on m , and H is the mean curvature vector of S in \mathbb{R}^l along f .*

The special Lagrangian submanifold $t \cdot L'^a$ is minimal (i.e. $H=0$) in Y' . Thus:

Corollary 3.4.3. [Sobolev inequality]. (Cf. [Jo3: III. Corollary 6.10].) *There exists $D_1 > 0$ such that $\|u\|_6 \leq D_1\|du\|_{L^2}$ for $t > 0$ and $u \in C_{cs}^1(t \cdot L'^a)$.*

Proposition 3.4.4. [Sobolev inequality]. (Cf. [Jo3: III Proposition 6.11].) *There exists $D_2 > 0$ such that for all $v \in C_{cs}^1(X^\diamond)$,*

$$\|v\|_{L^6} \leq D_2 \left(\|dv\|_{L^2} + \left| \int_{X^\diamond} v dV_g \right| \right).$$

Proof. Note that since X is connected, so is X^\diamond . The proof of [Jo3: III Proposition 6.11] then goes through, word for word, with only

- ‘embedding’ replaced by ‘immersion’;
- [Jo3: I. Theorem 2.17] replaced by Lemma 2.1.2 and Example 2.1.3 of the current notes to take care of the counter situation of branched coverings under study.

□

(Non)existence of A_7 in Statement 3.4.1.

Recall the parameters $tR'_0 < b_1^t < b_2^t (= Ct^{1-\eta}) < R_0 \leq 1$, which set the range of gluings in the gluing construction in Sec. 3.1 from the aspect of target Calabi-Yau 3-fold Y . $t \in (0, \delta)$ with $0 < \delta < 1$ small enough. As $f^t : X \rightarrow Y$ is an immersion with X carrying the pull-back metric, we'll directly treat X as a submanifold in Y in the following computations and estimates whenever this is more convenient.

Let $a, b \in \mathbb{R}$ with $0 < a < b < 1 - \eta$. Then for δ_0 small enough, $2b_2^t < t^b < t^a < R_0$ holds for all $t \in (0, \delta)$. Let $G : (0, \infty) \rightarrow [0, 1]$ be a smooth decreasing function with $G(s) = 1$ for $s \in (0, a]$ and $G(s) = 0$ for $s \in [b, \infty)$. Write \dot{G} for dG/ds . Recall the coordinates (x_1, x_2, x_3) on $\nu_X(\tilde{\Gamma}_j)$ and the coordinates (u_1, u_2, u_3) on $\nu_{Z_0}^j(\tilde{\Gamma}_i)$ in Definition 3.1.1. For $x \in \nu_X(\tilde{\Gamma}_j)$ for some $j \in \{1, \dots, \tilde{n}_0\}$, denote $r := |(x_1 + \sqrt{-1}x_2)^{m_j}|$. For $t \in (0, \delta)$, define a function $F^t : N^t = X \rightarrow [0, 1]$ by

$$F^t(x) = \begin{cases} 0, & \text{for } x \in \nu_X(\tilde{\Gamma}_j), j \in \{1, \dots, \tilde{n}_0\}, 0 \leq r \leq t^b, \\ G((\log r)/(\log t)), & \text{for } x \in \nu_X(\tilde{\Gamma}_j), j \in \{1, \dots, \tilde{n}_0\}, t^b \leq r \leq t^a, \\ 1, & \text{for } x \text{ at elsewhere in } X. \end{cases}$$

⁷Details that are the same as in [Jo3: III Sec. 6.4] and not needed in the discussion are referred to ibidem and, hence, omitted.

Then, F^t is a smooth function on N^t with

$$dF^t = \begin{cases} (\log t)^{-1} \dot{G}((\log r)/(\log t)) r^{-1} dr & \text{on } \coprod_{j=1}^{\tilde{n}_0} \{t^b \leq r \leq t^a\} \subset \coprod_{j=1}^{\tilde{n}_0} \nu_X(\tilde{\Gamma}_j), \\ 0 & \text{elsewhere.} \end{cases}$$

$\{F^t, 1 - F^t\}$ gives thus a two-component partition of unity on N^t . We'll use it to decompose a function $v \in C^1(N^t)$ with $\int_{N^t} v dV^t = 0$ into $v = F^t v + (1 - F^t)v$; and then treat $F^t v$ as a compactly-supported function on X^\diamond and apply Proposition 3.4.4 to it, and $(1 - F^t)v$ as a compactly supported function on $\coprod_{j=1}^{\tilde{n}_0} (t \cdot L^{a_j}) \subset \coprod_{j=1}^{\tilde{n}_0} Y'^j$ and apply Corollary 3.4.3 to each component of it.

For the first part, since f^t and f coincide on $X - \coprod_{j=1}^{\tilde{n}_0} \{0 \leq r \leq b_2^t\}$ and $b_2^t < t^b$, $F^t v$ is naturally in $C_{cs}^1(X^\diamond)$ and $g^t = g$ on the support of $F^t v$. Proposition 3.4.4 plus a Hölder's inequality gives then

$$\begin{aligned} \|F^t v\|_{L^6} &\leq D_2 \left(\|d(F^t v)\|_{L^2} + \left| \int_{N^t} F^t v dV^t \right| \right) \\ &\leq D_2 (\|F^t dv\|_{L^2} + \|v\|_{L^6} \cdot (\|dF^t\|_{L^3} + \|1 - F^t\|_{L^{6/5}})) . \end{aligned}$$

For the second part, we prove first a lemma:

Lemma 3.4.5. [quasi-isometry with uniform bound on dilatation]. *Assume that $b_1^t = C_1 t^{1-\eta_1}$ and $b_2^t = C_2 t^{1-\eta_2}$, where $0 < \eta_1 < \eta_2 < 1 - a < 1$. Assume further that $\eta_2 > \max_j \left\{ \frac{1}{2} \left(1 + \frac{1}{m_j}\right), \frac{1-a}{m_j} \right\}$. Then, for $\delta > 0$ small enough and all $t \in (0, \delta)$, the map*

$$f^t(x_1, x_2, x_3) \longmapsto ((x_1 + \sqrt{-1}x_2)^{m_j}, x_3, a^{1/m_j} t^{(m_j-1)/m_j} (x_1 + \sqrt{-1}x_2), 0)$$

gives a diffeomorphism that takes $f^t(\{|x_1 + \sqrt{-1}x_2|^{m_j} \leq t^a\})$ to $L'^{a_j} t^{m_j-1} \Big|_{|u_1 + \sqrt{-1}u_2| \leq t^a}$ quasi-isometrically with the dilatation uniformly bounded both from above and from below (away from zero) t -independently.

Proof. Recall Remark 3.1.6 and the proof of Definition/Lemma 3.3.1. Under the assumption that $b_1^t = C_1 t^{1-\eta_1}$ and $b_2^t = C_2 t^{1-\eta_2}$, where $0 < \eta_1 < \eta_2 < 1 - a < 1$, one has the following estimates over $(tR'_0 <) b_1^t \leq r \leq t^a$:

$$\begin{aligned} &\left| \frac{d}{dr} \left(a^{1/m_j} t^{(m_j-1)/m_j} \left[\frac{m_j}{m_j+1} \dot{\chi}^t(r) + \chi^t(r) r^{1/m_j} \right] \right) \right| \\ &= a^{1/m_j} t^{(m_j-1)/m_j} \left| \frac{m_j}{m_j+1} \ddot{\chi}^t(r) + \dot{\chi}^t(r) r^{1/m_j} + \frac{1}{m_j} \chi^t(r) r^{(1-m_j)/m_j} \right| \\ &\leq a^{1/m_j} t^{(m_j-1)/m_j} \left(\frac{m_j}{m_j+1} \cdot \frac{C_0}{(b_2^t)^2} + \frac{C_0}{b_2^t} r^{1/m_j} + \frac{1}{m_j} r^{(1-m_j)/m_j} \right) \\ &\leq O(t^{2\eta_2-1-\frac{1}{m_j}}) + O(t^{\eta_2-\frac{1-a}{m_j}}) + O(t^{(1-\frac{1}{m_j})\eta_1}). \end{aligned}$$

The further assumption that $\eta_2 > \max_j \left\{ \frac{1}{2} \left(1 + \frac{1}{m_j}\right), \frac{1-a}{m_j} \right\}$ implies that the right-hand side of the last inequality tends to zero as $t \rightarrow 0$. On the other hand,

$$\left| \frac{d}{dr} \left(a^{1/m_j} t^{(m_j-1)/m_j} r^{1/m_j} \right) \right| \leq O(t^{1-\frac{1-a}{m_j}})$$

over $b_1^t \leq r \leq t^a$, which also tends to 0 as $t \rightarrow 0$. The lemma follows. \square

Applying Corollary 3.4.3 to $(1 - F^t)v$ on $t \cdot L'^a$ gives $\|(1 - F^t)v\|_{L^6} \leq D_1 \|d((1 - F^t)v)\|_{L^2}$ with the norms computed using g' on $t \cdot L'^a$. Increasing D_1 to $2D_1$, the same inequality holds with norms computed using g^t for small t . Thus, making $\delta > 0$ smaller if necessary, for all $t \in (0, \delta)$ and each connected component of $(1 - F^t)v$ on N^t (labelled by $j = 1, \dots, \tilde{n}_0$),

$$\|(1 - F^t)v\|_{L^6} \leq 2D_1 \|d((1 - F^t)v)\|_{L^2}.$$

Summing over j gives

$$\begin{aligned} \|(1 - F^t)v\|_{L^6} &\leq 2\sqrt{\tilde{n}_0} D_1 \|d((1 - F^t)v)\|_{L^2} \\ &\leq 2\sqrt{\tilde{n}_0} D_1 (\|(1 - F^t)dv\|_{L^2} + \|v\|_{L^6} \cdot \|dF^t\|_{L^3}). \end{aligned}$$

Combining the above inequality with the previous inequality for $\|F^t v\|_{L^6}$, the inequalities $\|F^t dv\|_{L^2}$, $\|(1 - F^t)dv\|_{L^2} \leq \|dv\|_{L^2}$, and a Hölder's inequality then proves

$$\left[1 - (D_2 + 2\sqrt{\tilde{n}_0}D_1) \|dF^t\|_{L^3} - D_2 \|1 - F^t\|_{L^{6/5}}\right] \cdot \|v\|_{L^6} \leq (D_2 + 2\sqrt{\tilde{n}_0}D_1) \|dv\|_{L^2}.$$

It follows from Lemma 3.4.5 that one can compute the t -order of $\|dF^t\|_{L^3}$ and the t -order of $\|1 - F^t\|_{L^{6/5}}$ on (N^t, g^t) via the t -order of the corresponding same quantity on L' and $t \cdot L'^a$ in $\coprod_{j=1}^{\tilde{n}_0} (Y'^j, g')$ respectively.

On $L' \subset Y'^j$,

$$\begin{aligned} \|dF^t\|_{L^3} &= \left(\int_{u_3=0}^{l_j} \int_{\theta=0}^{2m_j\pi} \int_{r=t^b}^{t^a} \left| (\log t)^{-1} \dot{G}((\log r)/(\log t)) r^{-1} \right|^3 r dr d\theta du_3 \right)^{1/3} \\ &= l_j^{1/3} (2m_j\pi)^{1/3} (-\log t)^{-1} \left(\int_{r=t^b}^{t^a} |\dot{G}((\log r)/(\log t))|^3 r^{-2} dr \right)^{1/3}. \end{aligned}$$

It follows from the facts that $|\dot{G}((\log r)/(\log t))|$ is bounded above by a constant independent of t and r and that there exist $(0 <) a < a' < b' < b$ such that the restriction of $|\dot{G}((\log r)/(\log t))|$ to $[t^{b'}, t^{a'}]$ is bounded below by a positive constant independent of t and r that

$$O(-(\log t)^{-1} t^{-b'/3}) \leq \|dF^t\|_{L^3} \leq O(-(\log t)^{-1} t^{-b/3}).$$

Since $b, b' > 0$, it follows that $\|dF^t\|_{L^3} \rightarrow \infty$ as $t \rightarrow 0$.

On $t \cdot L'^{a_j} \subset Y'^j$,

$$\begin{aligned} \|1 - F^t\|_{L^{6/5}} &= \left(\int_{u_3=0}^{l_j} \int_{\theta=0}^{2m_j\pi} \int_{r=0}^{t^a} |1 - G((\log r)/(\log t))|^{6/5} \right. \\ &\quad \cdot r \left(1 + m_j^{-2} a_j^{2/m_j} r^{2(1-m_j)/m_j} \right) dr d\theta du_3 \Big)^{5/6} \\ &\leq l_j^{5/6} (2m_j\pi)^{5/6} \left(\frac{1}{2} \left[r^2 + m_j^{-1} a_j^{2/m_j} r^{2/m_j} \right]_0^{t^a} \right)^{5/6} \\ &= O\left(t^{5a/(3m_j)}\right) \longrightarrow 0 \quad \text{as } t \longrightarrow 0. \end{aligned}$$

It follows that one cannot extract the Sobolev inequality in Statement 3.4.1 that bounds $\|v\|_{L^6}$ by $\|dv\|_{L^2}$ from the inequality derived earlier: $[1 - (D_2 + 2\sqrt{\tilde{n}_0}D_1) \|dF^t\|_{L^3} - D_2 \|1 - F^t\|_{L^{6/5}}] \cdot \|v\|_{L^6} \leq (D_2 + 2\sqrt{\tilde{n}_0}D_1) \|dv\|_{L^2}$. The validity of Statement 3.4.1 is thus left open in our situation through the above method.

4 Summary and remark: Input from the topology of X and the branching of f .

Given a branched covering $f : X \rightarrow Z_0 \simeq S^3 \subset Y$ a special Lagrangian 3-sphere Z_0 in a Calabi-Yau 3-fold Y , one expects that the natural family of immersed Lagrangian deformations $f^t : X \rightarrow Y$ of f , $t \in (0, \delta)$ for $0 < \delta < 1$ small enough, constructed in Sec. 3.1 (1) have the Sobolev norms of the variations of the mean curvature along f^t too large as $t \rightarrow 0$, (cf. Sec. 3.2), (2) cannot pass directly a Sobolev inequality test that allows one to conclude the existence of a t -independent uniform positive lower bound for the first eigenvalue of the Laplacian Δ_{g^t} of (X, g^t) , where $g^t := (f^t)^*g$ is the pull-back metric on X , (cf. Sec. 3.4), though (3) f^t still meets the criteria on the size and the geometry in a Lagrangian neighborhood of f^t in Y that demand f^t to approach the singular (X, g^0) not too fast, as $t \rightarrow 0$, and to have enough room in Y to deform f^t further, (cf. Sec. 3.3). Issue (1) suggests that one has to glue the approximately special Lagrangian local models $a \cdot L'^a$ not to f , but some Lagrangian perturbation of f that matches the flaring-out rate of tL'^a better. Issue (2) suggests that one has to identify the collection of eigenvalues of (X, g^t) that approach 0 as $t \rightarrow 0$ and then consider deformations of f that are only in the complementary directions to those that correspond to the eigenfunction associated to these eigenvalues. Item (3) suggests that the gluing with a scaling construction, as done in [Jo3] of Joyce, is a versatile construction, which one would like to keep while remedying Issues (1) and (2).

While the detail of the deviation requires specific computations as is done in Sec. 3, that the immersed Lagrangian deformation constructed in this note will deviate from being deformable to an immersed special Lagrangian deformation, no matter what adjustments one attempts, is anticipated even before one gets into such details. This is because there are examples of branched coverings of S^3 by S^3 and the construction in this note apply to them as well. Should there be no deviations, one would have constructed a nontrivial family of immersed special Lagrangian S^3 's, which contradicts (the immersed generalization of) a result of Robert McLean ([McL: Theorem 3.6]) which implies, in particular, that an immersed special Lagrangian 3-sphere is rigid. Thus, if such a deformation is possible, the topology of the domain X of the branched covering $f : X \rightarrow S^3$ of a special Lagrangian S^3 in question must play a significant role in rectifying the deviations created in the naive gluing as we see in this note. The detail of the local model given in Sec. 2.2 combined with the original study of deformations of special Lagrangian submanifolds in [McL: Sec. 3] suggests that one such necessary condition would be:

- [infinitesimal condition for unwrapping] There exists a harmonic/closed-'n'-coclosed 1-form α on X that has the decay behavior exactly of type $\rho^{m_i}d\rho$ around the component Γ_i of the branch locus $\Gamma \subset X$ of the branched covering $f : X \rightarrow S^3$ in a Calabi-Yau 3-fold Y .

This is a condition that involves both the topology of X and the branching of the map $f : X \rightarrow S^3$. Here, m_i is the degree of f around Γ_i , ρ is the distance function $distance(\cdot, \Gamma)$ on X associated to the pull-back metric (with curvature singularities along Γ), and the harmonicity/closed-'n'-coclosedness of α is with respect to the latter singular metric on X as well. If α behaves of type $\rho^{\lambda_i}d\rho$ along Γ_i with $\lambda_i < m_i$ for some i , then one has a chance of tearing off f along Γ_i when trying to deform f as guided by α in the realm of special Lagrangian immersions. In the opposite direction, if $\lambda_i > m_i$ for some i , then the corresponding candidate special Lagrangian deformations of f may remain branched along Γ_i and hence fails to be an immersion.

This leads one thus to issues on such harmonic/closed-'n'-coclosed 1-forms and the existence/nonexistence of obstructions to the above first order picture.

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